Given two discrete imprecise random variables $X$ and $Y$, whose probability measures belong to credal sets $\mathcal{M}(\text{Bel}(X))$ and $\mathcal{M}(\text{Bel}(Y))$ respectively, and whose dependency is modeled by a copula $C$, how to join those two credal sets using the copula $C$?

**Problem statement**

We join every element of $\mathcal{M}(\text{Bel}(X))$ and $\mathcal{M}(\text{Bel}(Y))$ using the copula $C$ and take its lower bound $\mathcal{L}(\text{copula})$ on every event. This is the lower bound of the tightest credal set $M_{\text{robust}}$ containing every marginal probability joined by $C$. This is usually the optimal credal set one wishes to know, but it can be very hard to compute its lower bounds.

**Copulas**

A copula is a mapping $C : [0,1]^n \rightarrow [0,1]$ which can model any dependency between $n$ Cumulative Distribution Functions. Sklar’s theorem [3] states that any multivariate CDF $G$ can be expressed using a unique copula $C$ and its marginals $F_i$:

$$G(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)).$$

Conversely, any copula applied to $n$ univariate CDFs correctly defines a $n$-dimension CDF.

**Robust Method**

This method defines a bivariate belief function $\mathcal{L}_{\text{mass}}$ by joining the masses of the marginal belief functions using the $H$-volume of a copula [2]. In the bivariate case, writing $A_x = \sum m_X(a_i)$ and $B_x = \sum m_Y(b_i)$, the $H$-volume is:

$$m_X(a_i, b_j) = C(A_x, B_x).$$

The credal set resulting from $\mathcal{L}_{\text{mass}}$ is noted $M_{\text{mass}}$. There is in general no reason for having $M_{\text{robust}} = M_{\text{mass}}$. To compute the cumulated masses, it is necessary to specify an order on $C(A_x, B_x)$ using the copula $C$.

**Cumulated masses method**

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**Necessity functions as Marginals**

Possibility distributions define special types of belief functions called Necessity functions:

$$Nec(A) = 1 - sup_{\alpha \in A} \pi(x).$$

The focal sets of a Necessity function are its $\alpha$-cuts of its possibility distribution:

$$\pi(x) = \alpha \iff \pi(x) > \alpha.$$  

When marginals are necessity functions, it holds that $\mathcal{L}(\text{Nec}) = \mathcal{L}(\pi)$. But both $M_{\text{robust}} \subseteq M_{\text{mass}}$ and $M_{\text{robust}} \supseteq M_{\text{mass}}$ can happen.

**Product Copula**

When using the product copula (independence), it holds that $[1]: M_{\text{robust}} \subseteq M_{\text{mass}} \subseteq M_{\text{Copula}}$. Their lower envelope coincide on Cartesian product of events.

**Conclusion and Perspectives**

- Three methods for joining copulas and credal sets: depending on the marginals, some inclusions allow for easier computations.
- Work done since submission: If the copula is directionally convex, it holds that $\mathcal{L}(\text{Copula}) = \mathcal{L}(\text{Nec})$ and thus $M_{\text{robust}} \subseteq M_{\text{mass}}$.
- Future work with imprecise copulas or other imprecise models (clouds)
- Finding alternative orders for Necessity functions (seems an order for $M_{\text{robust}} \subseteq M_{\text{Copula}}$ always exist)

**Acknowledgment**

This project has received financial support from the CNRS through the MITI interdisciplinary programs and from CNES.

**References**

