A correspondence between fuzzy orthopartitions and credal partitions

Stefania Boffa and Davide Ciucci

University of Milano-Bicocca, Italy

BELIEF22

October 28, 2022

Main Objectives

- **Fuzzy orthopartitions $\leftrightarrow$ Credal partitions**
- **They are generalizations of standard partitions;**
- **They are used to represent partitions in case of partial knowledge on the membership class of the elements.**

Credal partitions

- Let $U = \{u_1, \ldots, u_l\}$ be a universe;
- let $C = \{C_1, \ldots, C_n\}$ be a standard partition of U.

A **credal partition** is a collection $m = \{m_1, \ldots, m_l\}$ of basic belief assignments.

Thierry Denoeux and Marie-Hélène Masson. Evclus: evidential clustering of proximity data. IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics), 34(1):95–109, 2004.

Normalized basic belief assignments

$m_1, \ldots, m_l$ are **normalized**

Example:

A	$m_1(A)$	$m_2(A)$	$m_3(A)$
$\emptyset$	0	0	0
C_1	0.3	0.4	0
C_2	0.4	0.5	0.4
$\{C_1, C_2\}$	0.3	0.1	0.6

Intuitionistic fuzzy sets

Fuzzy orthopartitions are collections of intuitionistic fuzzy sets.

Definition:
An **intuitionistic fuzzy set** A of a universe U is defined as

$$A = (\mu, \nu)$$

where the maps $\mu : U \to [0, 1]$ and $\nu : U \to [0, 1]$ satisfying the following condition: for each $u \in U$

$$\mu(u) + \nu(u) \leq 1.$$

Interpretation for fuzzy orthopartitions

We attach the following semantics to a fuzzy orthopartition.

- $(\mu_1, \nu_1), \ldots, (\mu_n, \nu_n)$ respectively correspond to $C_1, \ldots, C_n$;
- Let (μ_i, ν_i) be an IFS on U,

  $\mu_i(u)$ is the **degree of belief** that **"u belongs to C_i"**;
  $\nu_i(u)$ is the **degree of belief** that **"u does not belong to C_i"**;
  $pl_i(u) = 1 - \nu_i(u)$ is the **degree of plausibility** that **"u belongs to C_i"**.

Fuzzy orthopartitions

Definition:
$\mathcal{O} = \{(\mu_1, \nu_1), \ldots, (\mu_n, \nu_n)\}$ is a **fuzzy orthopartition** of U if and only if for each $u \in U$:

1. $\sum_{i=1}^{n} \mu_i(u) \leq 1$ (**disjoint blocks**),
2. $\mu_i(u) + (pl_j(u) - \mu_j(u)) \leq 1$, for each $i \neq j$ (**disjoint blocks**),
3. $\sum_{i=1}^{n} pl_i(u) \geq 1$ (**covering condition**),
4. for each $i \in \{1, \ldots, n\}$ with $pl_i(u) - \mu_i(u) > 0$, there exists $j \in \{1, \ldots, n\} \setminus \{i\}$ such that $pl_j(u) - \mu_j(u) > 0$ (**the uncertainty cannot regard only one block**).

Fuzzy orthopartitions

Fuzzy orthopartitions are generalizations of

- **standard orthopartitions**
- **Fuzzy probabilistic partitions**

Classical orthopartitions

An **orthopartition** is a generalized partition where blocks are **orthopairs**, namely **pairs of disjoint subsets** of the initial universe.

Example:
$C_1 = (\{a, b\}, \{c\})$ and $C_2 = (\{c\}, \{a, b\})$ form an orthopartition of $\{a, b, c, d\}$.

$$a, b \in C_1, \quad c \in C_2, \quad d \in \textbf{?}.$$

O is understood as a partition where

- the **membership class of some elements is known with certainty**,
- whereas **the membership class of the remaining elements is completely unknown**.

Fuzzy orthopartitions and fuzzy probabilistic partitions

A fuzzy orthopartition where all degrees of belief and plausibility coincide, specifies a fuzzy probabilistic partition.

Fuzzy orthopartition with $\mu_i(u) = pl_i(u)$

u	$\mu_1(u)$	$\nu_1(u)$	$pl_1(u)$	$\mu_2(u)$	$\nu_2(u)$	$pl_2(u)$
u_1	0.2	0.8	0.2	0.8	0.2	0.8
u_2	0.5	0.5	0.5	0.5	0.5	0.5
u_3	0.6	0.4	0.6	0.4	0.6	0.4

Fuzzy probabilistic partition

u	C_1	C_2
u_1	0.2	0.8
u_2	0.5	0.5
u_3	0.6	0.4

Credal partitions and Fuzzy probabilistic partitions

- Credal partitions subsume the concept of **fuzzy probabilistic partitions**.
- $m = \{m_1, \ldots, m_l\}$ is a fuzzy probabilistic partition iff $m_1, \ldots, m_l$ are **Bayesian bbas** ($m_i(A) = 0$ for each $|A| > 1$).

A	$m_1(A)$	$m_2(A)$	$m_3(A)$
$\emptyset$	0	0	0
C_1	0.1	0.2	0.4
C_2	0.8	0.5	0.4
C_3	0.1	0.3	0.2
$\{C_1, C_2\}$	0	0	0
$\{C_2, C_3\}$	0	0	0
$\{C_1, C_3\}$	0	0	0
$\{C_1, C_2, C_3\}$	0	0	0

A first correspondence

Fuzzy orthopartitions and credal partitions coincide when both are fuzzy probabilistic partitions.

Fuzzy orthopartition

	u_1	u_2	u_3
$\mu_1(u) = pl_1(u)$	0.2	0.5	0.6
$\mu_2(u) = pl_2(u)$	0.8	0.5	0.4

Credal partition

A	$m_1(A)$	$m_2(A)$	$m_3(A)$
$\emptyset$	0	0	0
C_1	0.2	0.5	0.6
C_2	0.8	0.5	0.4
C	0	0	0

An extended correspondence

We consider

- $\mathcal{C}_O$: the set of all fuzzy probabilistic partitions **compatible** with O;
- $\mathcal{C}_m$: the set of all fuzzy probabilistic partitions **compatible** with m.

In a dynamic situation, where the **knowledge** about the membership class of the elements

- **is partial** and
- **increases (for example over the time)**

so that
fuzzy orthopartitions and credal partitions become fuzzy probabilistic partitions.

Compatible fuzzy probabilistic partitions

A fuzzy probabilistic partition π is **compatible** with m iff

$$m_j(\{C_i\}) \leq \pi_j(C_i) \leq \sum_{\{A \mid C_i \in A\}} m_j(A).$$

Credal partition $m = (m_1, m_2, m_3)$ of $\{u_1, u_2, u_3\}$

A	$m_1(A)$	$m_2(A)$	$m_3(A)$
$\emptyset$	0	0	0
C_1	0.3	0.4	0
C_2	0.4	0.5	0.4
$\{C_1, C_2\}$	0.3	0.1	0.6

Fuzzy probabilistic partition π compatible with m

$m_1(C_1)$	$\pi_1(C_1)$	$m_1(C_1) + m_1(\{C_1, C_2\})$
0.3	0.4	0.6

Compatible fuzzy probabilistic partitions

A fuzzy probabilistic partition π is **compatible** with O iff

$$\mu_i(u_j) \leq \pi_j(C_i) \leq pl_i(u_j).$$

$\pi = \{\pi_1, \pi_2\}$ **is compatible with** $O = (\mu_1, \nu_1), (\mu_2, \nu_2)$

$\mu_1(u)$	$\pi_1(u) = u$ belongs to C_1	$pl_1(u)$
0.2	0.4	0.5

$\mu_2(u)$	$\pi_2(u) = u$ belongs to C_2	$pl_2(u)$
0.5	0.6	0.6

From fuzzy orthopartitions to credal partitions

$O=$ fuzzy orthopartition $\rightarrow$ $\mathcal{F}(O)=$ class of credal partitions

$m=(m_1,\ldots,m_l)\in\mathcal{F}(O) \leftrightarrow$ 1) $m_j(\{C_i\})=\mu_i(u_j)$;
2) $\sum_{\{A \mid C_i\in A\}} m_j(A)=pl_i(u_j)$.

Fuzzy orthopartition $O=\{(\mu_1,\nu_1),(\mu_2,\nu_2),(\mu_3,\nu_3),(\mu_4,\nu_4)\}$

$\mu_1(u)$	$\nu_1(u)$	$\mu_2(u)$	$\nu_2(u)$	$\mu_3(u)$	$\nu_3(u)$	$\mu_4(u)$	$\nu_4(u)$
0.1	0.5	0.1	0.7	0.25	0.45	0.15	0.55

Basic Belief Assignment

$$m(A)=\begin{cases} 0.1 & \text{if } A\in\{\{C_1\},\{C_2\},\{C_1,C_2,C_3\},\{C_1,C_2,C_4\}\},\\ 0.2 & \text{if } A=\{C_1,C_3,C_4\},\\ 0.25 & \text{if } A=\{C_3\},\\ 0.15 & \text{if } A=\{C_4\},\\ 0 & \text{otherwise.}\end{cases}$$

$m(\{C_1\})+m(\{C_1,C_2,C_3\})+m(\{C_1,C_2,C_4\})+m(\{C_1,C_3,C_4\})=0.1+0.1+0.1+0.2=0.5=pl_1(u)$.

From fuzzy orthopartitions to credal partitions

Theorem

Let $m \in \mathcal{F}(O)$ if and only if m has the same compatible partitions of O, namely $\mathcal{C}_m = \mathcal{C}_O$.

Theorem

Let O be a fuzzy orthopartition of U, then

$$\mathcal{F}(O) = \begin{cases} \emptyset \\ 1 \\ infinite \end{cases}$$

From fuzzy orthopartitions to credal partitions

Let $u_j \in U$, a bba m_j is determined by solving

$$S_j = \begin{cases} \mu_1(u_j) + \sum_{\{A \mid \{C_1\} \subset A\}} x_A^1 = pl_1(u_j), \\ \vdots \\ \mu_n(u_j) + \sum_{\{A \mid \{C_n\} \subset A\}} x_A^n = pl_n(u_j), \\ \mu_1(u_j) + \ldots + \mu_n(u_j) + \sum_{x \in \mathcal{A}} x = 1, \end{cases}$$

S_j is a linear system with $n+1$ equations and $2^n - n - 1$ variables.

- each variable corresponds to $m_j(A)$ where $|A| \geq 2$;
- the first n equations assure that the total of the masses of C_i is the plausibility of C_i;
- the last equation allow m_j to be a bba.

From fuzzy orthopartitions to credal partitions

- $m = (m_1, \ldots, m_l) \in \mathcal{F}(O)$;
- $m_1, \ldots, m_l$ are determined by solving $S_1, \ldots, S_l$;
- $S_1, \ldots, S_l$ can admit zero, one or infinite solutions.

Then,

- $\mathcal{F}(O) = \emptyset$ iff there exists u_j such that S_j is impossible.
- $|\mathcal{F}(O)| = \infty$ iff $S_1, \ldots, S_l$ are non-impossible and at least one of them is indeterminate.
- $|\mathcal{F}(O)| = 1$ iff $S_1, \ldots, S_l$ are determinate.

From a credal partition to an orthopartition

Let $m = \{m_1, \ldots, m_l\}$ be a credal partition. Then,

$O_m = \{(\mu_1, \nu_1), \ldots, (\mu_n, \nu_n)\}$ is an orthopartition,

where μ_j and pl_j are defined by

- $\mu_j(u_i) = m_i(C_j)$;
- $pl_j(u_i) = \sum_{\{A \mid C_j \in A\}} m_i(A)$.

Theorem

O_m is a fuzzy orthopartition of U verifying $\mathcal{C}_{O_m} = \mathcal{C}_m$.

- Different credal partitions can correspond to a same fuzzy orthopartition.
- Therefore, we can define the following equivalence relation

$$m \sim m' \text{ if and only if } O_m = O_{m'}.$$

Theorem

Let O be a fuzzy orthopartition of U and let $m \in \mathcal{F}(O)$. Then, $O_m = O$.

$$O \to \mathcal{F}(O) \to O_m$$

Conclusions and Future directions

- Given $m \rightarrow$ there exists O_m such that $\mathcal{C}_m = \mathcal{C}_{O_m}$;
- When $\mathcal{F}(O) = \emptyset$,
  $\nexists$ a credal partition m such that $\mathcal{C}_O = \mathcal{C}_m$.

We want to extended our results in the more general case of no-normalized bbas. This will enable us to deal with the presence of outliers in clustering applications.

There are two possible directions:

- relaxing the axioms of fuzzy orthopartitions;
- adding another intuitionistic fuzzy set (μ_0, ν_0) to the initial fuzzy orthopartition.

Thanks for the attention!