Toward updating belief functions over Belnap–Dunn logic

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Belief 2022

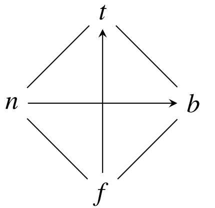
Goal

To study updating of belief and plausibility functions in the presence of contradictory information

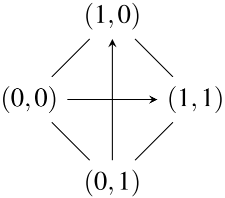
1. Representing incomplete/contradictory probabilistic information
    - Belnap-Dunn Logic
    - Non-standard probabilities
2. Belnap–Dunn models
3. updating belief and plausibility in BD-frames

Belnap-Dunn logic 1

Language L_{BD}: $\varphi := p \in \text{Prop} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg\varphi$



Belnap-Dunn square 4



Belnap-Dunn square $(\mathbf{4}, \wedge, \vee, \neg)$ is a de Morgan algebra.

- $(\mathbf{4}, \wedge, \vee)$ is a lattice
- each element represents the available positive and/or negative information
  - n: no information
  - f: false (bottom)
  - t: true (top)
  - b: contradictory information

Belnap-Dunn Logic: Models

Language $L_{\mathsf{BD}}, \varphi := p \in \texttt{Prop} \mid \varphi \wedge \varphi \mid \varphi \wedge \varphi \mid \neg\varphi$

Frame model and frame semantics

$\mathfrak{M} = \langle S,\ v^+, v^- \ : \texttt{Prop} \to \mathscr{P}(S) \rangle$ where v^+ and v^- are extended in the standard way and $[\![]\!]^+$ and $[\![]\!]^-$ denote extensions of v^+ and v^-. Let $\varphi, \varphi' \in \mathscr{L}_{\mathsf{BD}}$. For a model $\mathfrak{M} = \langle S, v^+, v^- \rangle$, we define $s \vDash^+ \varphi$ and $s \vDash^- \varphi$ for $s \in S$ as follows.

$$s \vDash^+ p \text{ iff } s \in v^+(p) \qquad s \vDash^- p \text{ iff } s \in v^-(p)$$
$$s \vDash^+ \neg\varphi \text{ iff } s \vDash^- \varphi \qquad s \vDash^- \neg\varphi \text{ iff } s \vDash^+ \varphi$$
$$s \vDash^+ \varphi \wedge \varphi' \text{ iff } s \vDash^+ \varphi \text{ and } s \vDash^+ \varphi' \qquad s \vDash^- \varphi \wedge \varphi' \text{ iff } s \vDash^- \varphi \text{ or } s \vDash^- \varphi'$$
$$s \vDash^+ \varphi \vee \varphi' \text{ iff } s \vDash^+ \varphi \text{ or } s \vDash^+ \varphi' \qquad s \vDash^- \varphi \vee \varphi' \text{ iff } s \vDash^- \varphi \text{ and } s \vDash^- \varphi'$$

Lindenbaum algebra

- Lindenbaum algebra : the set of equivalence classes of the formulas.
- Lindenbaum algebra of Belnap–Dunn logic is a De Morgan algebra. $\neg$ is an involutive de Morgan negation.

Non-standard probabilities

what are Non-standard probabilities?

- **Non-standard probabilities** : probabilistic extension of BD logic: Frame semantics [Klein, Majer, Rafiee-Rad 2020]
- **Idea**: to extend BD models with a probability measure. (independence of positive and negative (probabilistic) information)

A **probabilistic BD model** is a tuple $M = \langle S, v^+, v^-, \mu \rangle$, s.t.

- $\langle S, v^+, v^- \rangle$ is a BD model and
- $\mu : \mathscr{P}(S) \to [0,1]$ is a probability measure on S

properties

- Generate two maps $p^+, p^- : L_{BD} \to [0,1]$
- **Positive probability** of φ is the measure of its positive extension: $p^+(\varphi) := \mu([\![\varphi]\!]^+)$.
- Negative probability of φ: $p^-(\varphi) := \mu([\![\varphi]\!]^-)$.

we focus on one component: $p^-(\varphi) = p^+(\neg\varphi)$

Non-standard probabilities: axioms

[Klein et al] Lemma 1

Let $M = \langle S, v^+, v^-, \mu \rangle$ be a probabilistic BD model. Then the non-standard probability function p^+ induced by m satisfies:

(A1) normalization $\quad 0 \leq p^+(\varphi) \leq 1$

(A2) monotonicity $\quad$ if $\varphi \vdash_{BD} \psi$ then $p^+(\varphi) \leq p^+(\psi)$

(A3) inclusion-exclusion $\quad p^+(\varphi \wedge \psi) + p^+(\varphi \vee \psi) = p^+(\varphi) + p^+(\psi)$.

properties

- In general $p^+(\neg\varphi) \neq 1 - p^+(\varphi)$
- inconsistent information: one can have $0 < p^+(\varphi \wedge \neg\varphi)$
- incomplete information: $1 > p^+(\varphi \vee \neg\varphi)$

Models

Definition (DS models and their associated belief functions)

Let $\mathscr{L}_{\mathsf{BD}}$ be the Lindenbaum algebra for BD logic over the set of propositional letters $\mathtt{Prop}$. A DS *model* is a tuple $\mathscr{M} = \langle S, \mathscr{P}(S), \mathtt{Bel}, v^+, v^- \rangle$ such that $\langle S, v^+, v^- \rangle$ is a BD model and $\mathtt{Bel}$ is a belief function on $\mathscr{P}(S)$. We denote $\mathsf{bel}^+_{\mathscr{M}} : \mathscr{L}_{\mathsf{BD}} \to [0,1]$ and $\mathsf{bel}^-_{\mathscr{M}} : \mathscr{L}^{op}_{\mathsf{BD}} \to [0,1]$ the maps such that, for every $\varphi \in \mathscr{L}_{\mathsf{BD}}$,

$$\mathsf{bel}^+_{\mathscr{M}}(\varphi) = \mathtt{Bel}(|\varphi|^+) \qquad \text{and} \qquad \mathsf{bel}^-_{\mathscr{M}}(\varphi) = \mathtt{Bel}(|\varphi|^-) = \mathtt{Bel}(|\neg\varphi|^+). \tag{1}$$

Definition ($\mathsf{DS}_{\mathtt{pl}}$ models and their associated plausibility functions)

Let $\mathscr{L}_{\mathsf{BD}}$ be the Lindenbaum algebra for BD logic over the set of propositional letters $\mathtt{Prop}$. A $\mathsf{DS}_{\mathtt{pl}}$ *model* is a tuple $\mathscr{M} = (S, \mathscr{P}(S), \mathtt{Bel}, \mathtt{Pl}, v^+, v^-)$ such that $(S, \mathscr{P}(S), \mathtt{Bel}, v^+, v^-)$ is a DS model, $\mathtt{Pl}$ is a plausibility function on $\mathscr{P}(S)$. We denote $\mathtt{pl}^+_{\mathscr{M}} : \mathscr{L}_{\mathsf{BD}} \to [0,1]$ and $\mathtt{pl}^-_{\mathscr{M}} : \mathscr{L}^{op}_{\mathsf{BD}} \to [0,1]$ the maps such that, for every $\varphi \in \mathscr{L}_{\mathsf{BD}}$,

$$\mathtt{pl}^+_{\mathscr{M}}(\varphi) = \mathtt{Pl}(|\varphi|^+) \qquad \text{and} \qquad \mathtt{pl}^-_{\mathscr{M}}(\varphi) = \mathtt{Pl}(|\varphi|^-) = \mathtt{Pl}(|\neg\varphi|^+). \tag{2}$$

calssical updating methods

methods for updating beleif and plausibility

- Conditioning belief and plausibility via Demspter–Shafer combination rule (**DS-conditioning** denoted by $\mathtt{Bel}^B$): $\mathsf{m}_{\mathtt{Bel}} \oplus \mathsf{m}_B$
- Conditioning belief as **lower measure** and plausibility as **upper measure** denoted by $\mathtt{Bel}_B$: $\mathtt{Bel} = (\mathscr{M}_{\mathtt{Bel}})_*$(**Bayesian update** of every probability in $\mathscr{M}_{\mathtt{Bel}}$)

Modularity of Belnap–Dunn frame

allows applying the results in classical case such as classical updating belief and plausibility on the frame without requiring and searching for their counterparts on the De Morgan algebra

Unanswered questions and future works

obstacles

- **Not able to find a duality** : A counter example which shows that m_{bel} and $m_{\mathtt{Bel}}$ are not interchangeable
- **DS-conditioning**: definable on the algebra but different
- **Lower measure case** : its definition is an open question
- probability conditioning on distributive lattices is possible using **congruence lattices** but negation is problematic

Thank you for your attention!