

Nonholonomic Interpolation: a general methodology for motion planning in robotics

B. Berret*, J.P. Gauthier⁺ and V. Zakalyukin[†]

Abstract—In this paper, we present several constructive results about nonholonomic interpolation, in the perspective of motion planning in robotics. We specially treat the case of a set of nonholonomic constraints of corank $p \leq 3$. In fact, we are able to treat almost all generic cases for $p \leq 3$. But also, we show what may happen for larger corank. We give complete details in the Engel's case, which, from the point of view of robotics, corresponds to the kinematic constraints of a car with a trailer.

Keywords—motion planning, subriemannian geometry, robotics.

I. INTRODUCTION

We are given a motion planning problem, i.e. $\Sigma = (\Delta, g, \Gamma)$ where $\Gamma : [0; T_\Gamma] \rightarrow \mathbb{R}^n$ is a compact parametrized curve on \mathbb{R}^n , Δ is a completely non integrable distribution on \mathbb{R}^n of corank p , g is a riemannian metric over Δ . Then, (Δ, g) is a subriemannian metric over \mathbb{R}^n , the associated subriemannian distance being denoted by d . In practice, Δ is a set of nonholonomic constraints of a robot, g , the metric, measures the length of the admissible curves. Γ is a non-admissible path, that has been chosen in order to connect a source and a target in the ambient space \mathbb{R}^n , avoiding some obstacles.

The problem is to approximate the non-admissible motion Γ by an admissible one, the approximation being made in the subriemannian sense.

Typical dynamics Δ are the one of the *unicyle* and the *car with a trailer*:

$$\left\{ \begin{array}{l} \Delta_1 : \begin{cases} \dot{x} = \cos(\theta) u \\ \dot{y} = \sin(\theta) u \\ \dot{\theta} = v \end{cases} \\ \Delta_2 : \begin{cases} \dot{x} = \cos(\theta) v \\ \dot{y} = \sin(\theta) v \\ \dot{\theta} = u \\ \dot{\phi} = u - \sin(\phi) v \end{cases} \end{array} \right. \quad (1)$$

In both cases, the subriemannian length of the admissible path is, for instance:

$$l(\gamma) = \int_0^{T_\gamma} \|\dot{\gamma}\|_g dt = \int_0^{T_\gamma} \sqrt{u^2 + v^2} dt \quad (2)$$

* B. Berret is with the Motricity-Plasticity Laboratory, INSERM ERM207, University of Bourgogne, BAT. SCIENCES DU SPORT, BP 27877, 21078, DIJON CEDEX, France. bastien.berret@u-bourgogne.fr

+ J.P. Gauthier is with LE2I, UMR CNRS 5158, University of Bourgogne, BAT. MIRANDE, BP 47870, 21078, DIJON CEDEX, France. gauthier@u-bourgogne.fr

† V. Zakalyukin is with Moscow State University, 119 993, Leninski Gori, 1, Moscow, Russia. zakalyu@mail.ru

In fact, and this is one of the main conclusions of the paper, our results are in some sense, independant of the choice of the metric g .

Following Jean ([1], [2], [3]), we define the *metric-complexity* $MC(\Sigma, \varepsilon)$ and the *entropy* $E(\Sigma, \varepsilon)$ of the motion planning problem Σ , as follows. Here ε is a small parameter.

- 1) The *metric-complexity* $MC(\Sigma, \varepsilon)$ is $\frac{1}{\varepsilon}$ times the minimum length of the admissible curves that are at distance less than or equal to ε from Γ , and that connect the endpoints of Γ .
- 2) The (Kolmogorov's) *entropy* of Γ is $\frac{1}{\varepsilon}$ times the minimum length of the admissible curves γ that ε -interpolate the curves Γ (i.e., any piece of γ of length $\geq \varepsilon$ contains a point of Γ), and that connect the endpoints of Γ .

It follows from our paper [4] that, provided that ε is small enough, ε -interpolating curves minimizing the entropy *do exist*.

Our methods are constructive, i.e., in both cases (entropy and metric-complexity), we are able to construct *asymptotic optimal syntheses*, i.e., one parameter families (depending on ε) of admissible curves, $\gamma_\varepsilon : [0, T_{\gamma_\varepsilon}] \rightarrow \mathbb{R}^n$ that realize an *equivalent* of the entropy function or of the metric-complexity function, i.e., if l denotes the subriemannian length:

$$\left\{ \begin{array}{l} \lim_{\varepsilon \rightarrow 0} \frac{\frac{1}{\varepsilon} l(\gamma_\varepsilon)}{MC(\Sigma, \varepsilon)} = 1, \text{ or} \\ \lim_{\varepsilon \rightarrow 0} \frac{\frac{1}{\varepsilon} l(\gamma_\varepsilon)}{E(\Sigma, \varepsilon)} = 1. \end{array} \right. \quad (3)$$

The main result, on which we want to focus in this paper is the following (more or less philosophical point): in most cases, these asymptotic optimal syntheses are extremely robust. In fact, they are really stable, in the sense that, in certain *normal* coordinates, all of them are diffeomorphic, and in particular, they do not depend on the metric and on the curve Γ : they depend only on the integers appearing in the flag of brackets of Δ .

In the paper, we will just show the complete picture of what happens up to corank $p \leq 3$ for Δ .

For $p \geq 3$, some much more complicated phenomena appear, that we have started to elucidate in our papers [4], [10].

To say just a word, up to corank 8, the geometry of quaternions comes deeply inside the problem.

The paper is organized as follows.

In the next section (II), we define the crucial notions of normal coordinates and *nilpotent approximation along* Γ . We give a theorem (2.3) stating that the entropy and the metric complexity are equivalent to those of the nilpotent approximations along Γ .

In sections (III, IV, V) we give the complete picture of what happens for entropy and metric complexity up to $p \leq 3$. We show the corresponding *asymptotic optimal syntheses*.

Section (IV) contains also details about the Engel's case, i.e. the case of Δ_2 in (1): the car with a trailer.

In section (V), we give also a few explanations of what happens in the coranks 4-5 cases, the really wild cases starting from corank 6 on.

Of course, in this study, we are interested only with the generic cases. Here, the topology is the usual C^∞ topology over compact sets, since the problems restrict to ε -neighborhoods of the compact curve Γ .

Also, we will not consider the case (generic in some dimensions) where Γ is tangent to Δ at some isolated points. Referring to our previous papers, it is clear that the formulas do not change: some invariants tend to infinity at these points, but the integral formulas for $MC(\Sigma, \varepsilon)$ and $E(\Sigma, \varepsilon)$ are convergent.

Comments:

- 1) To summarize, in the paper, we try to present a synthesis of the results obtained in the papers [7], [8], [9], [10]. The details of the proofs of the new results in this paper will appear in [4] and [11]. These new results are mostly the relation $E = 2\pi MC$ between complexity and entropy, and the case of engel distribution.
- 2) Of course, there are other approaches to the motion planning problem treated here. See for instance [12],[13] and [14]. Our point of view starts mostly with the work of Jean ([1], [2], [3]), who first considered the concepts of metric complexity and entropy. What we do more in our approach is to give precise estimates for these quantities (Jean gave only the order of the leading term). Also we provide explicit constructions of corresponding optimal syntheses.
- 3) Our main conclusion, in view of results, is the following: our methodology and the constructions that follow are certainly highly recommandable for several reasons:
 - they are optimal, if some SR metric is specified.
 - they are (more than robust) STABLE in some sense. In particular, the optimal syntheses depend neither on the robot nor on the metric, but on the structure of the flag of brackets only. What is remarkable is that this robustness is generic up to corank 3, but fails to be true for higher corank (as will be shown in [4]).
 - Optimal syntheses are explicit, simple and easy to calculate.

II. NORMAL COORDINATES, NORMAL FORM, NILPOTENT APPROXIMATION ALONG Γ , EQUIVALENCE TO NILPOTENT APPROXIMATION

We are given a p -dimensional **parametrized** surface $S(u)$ transversal to Δ and containing Γ , which is itself assumed to be never tangent to Δ .

The normal coordinates w.r.t. S that we define here are a generalization of the geodesic coordinates in Riemannian geometry.

Theorem 2.1: (Normal coordinates) There is a germ¹ of global coordinates along Γ , $(x, y, w) = \xi$, such that :

- $\Gamma(t) = (0, 0, t)$, $S = \{(x, y, w) \mid x = 0\}$,
 $S(u_1, \dots, u_p) = (0, u_1, \dots, u_{p-1}, u_p)$;
- $\Delta_{\Gamma(t)} = \text{Ker } dw \cap_{i=1, \dots, p-1} \text{Ker } dy_i$, $g|_S = \sum_{i=1}^{n-p} dx_i^2$;
- The ε -cylinders $C_\varepsilon^S = \{\xi \mid d(\xi, S) = \varepsilon\}$ are the sets:

$$\{\xi \mid \sum_{i=1}^{n-p} x_i^2 = \varepsilon^2\}$$

Also, of course, the tubes

$$T_\varepsilon^S = \{\xi \mid d(\xi, S) \leq \varepsilon\} = \{\xi \mid \sum_{i=1}^{n-p} x_i^2 \leq \varepsilon^2\}. \quad (4)$$

Normal coordinates are not unique: changes of normal coordinates are given by transformations of the following type:

$$\tilde{x} = T(y, w)x, \quad (\tilde{y}, \tilde{w}) = (y, w) \quad (5)$$

Theorem 2.2: (Normal form w.r.t. normal coordinates) A parametrized surface $S(u)$ being given and a normal coordinate system $\xi = (x, y, w)$ w.r.t. $S(u)$ being chosen, there is a unique orthonormal frame (F_1, \dots, F_{n-p}) for the metric g , such that :

$$F_j = \sum_{i=1}^{n-p} Q_{ij}(x, y, w) \frac{\partial}{\partial x_i} + \sum_{i=1}^{p-1} L_{ij}(x, y, w) \frac{\partial}{\partial y_i} + M_j(x, y, w) \frac{\partial}{\partial w}$$

with:

- 1) The matrix $Q = (Q_{ij})$ is symmetric, $Q(0, y, w) = Id$,
- 2) ${}^2Q(x, y, w).x = x$
- 3) $L(x, y, w).x = 0$

Conversely, if we have an orthonormal frame F in coordinates ξ with properties 1), 2), 3), then, ξ are normal coordinates w.r.t. the parametrized surface $(0, y, w)$.

Examples:

- 1) $(n=4, p=2)$ The car with a trailer:

¹ As usual, it is an equivalence class of coordinates on a neighborhood of Γ , under the relation of agreeing on some subneighborhood of Γ .

² We denote by \cdot the usual product of a matrix by a vector, in normal coordinates.

In that case, we have generically a canonical choice of a parametrized surface S transversal to Δ :

The abnormal flow H (length 1) is a well defined vector field (up to sign). It has, up to sign, a length one orthonormal vector field I in the distribution Δ . Set $K = [I, H]$. Generically K is never tangent to Γ and never belongs to Δ . So we have a canonical 3-frame (I, H, K) that defines a subriemannian metric (Δ', g') over \mathbb{R}^4 of corank 1.

Again, we do not consider the case (generic) of isolated points where Δ' is tangent to Γ : they do not change anything in the final results. In that case, the surface $S(u_1, u_2)$ is given by:

$$S(u_1, u_2) = \exp(u_1 \cdot K(\Gamma(u_2))) \quad (6)$$

2) $(n=3, p=1)$ The unicycle:

In that case, the surface S is restricted to the curve Γ (it is one-dimensional).

It has been shown in [5], [6], that the normal form in normal coordinates $\xi = (x, y, w)$, is:

$$\begin{cases} \dot{x} &= (1 + y^2\beta) - xy\beta v \\ \dot{y} &= xy\beta u + (1 + x^2\beta)v \\ \dot{w} &= \frac{y}{2}\gamma u - \frac{x}{2}\gamma v \end{cases} \quad (7)$$

Here, u and v are the controls, and β, γ are smooth functions of (x, y, w) .

If $\gamma(0, 0, w) = 0$ (isolated points), then, the distribution Δ is *Martinet* at this point.

Now we explain what will be our *nilpotent approximation along* Γ .

We will explain this in the case where the flag of brackets does not change the dimension at isolated points (such as *Martinet* above), these cases are treated separately. Also, besides these cases, we will consider at most cases when we need brackets of order 2 to generate the whole tangent space (Engel for instance).

For our purpose $p \leq 3$, this will be enough. We give to the variables x_i the weight 1, to the variables y_i the weight 2, and to the variable w the weight zero. Remind that w is the parameter along the curve Γ .

Then, the vectorfield $\frac{\partial}{\partial x_i}$ gets weight -1, $\frac{\partial}{\partial y_i}$ gets weight -2, and (in order to agree with some *local effect* in the direction of Γ), we set that $\frac{\partial}{\partial w}$ has weight -2 in the one-step-bracket-generating case, and -3 in the remaining case (Engel)³.

³ Usually one proceeds as follows: there is a gradation on formal power series on the variables x, y, z with some natural weights. Differential operators $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial w}$ operate on formal power series, decreasing the degree. If ξ_i has order k , $\frac{\partial}{\partial \xi_i}$ has order $-k$, as an operator. The formal vector fields get also a gradation. For instance $f(\xi) \frac{\partial}{\partial \xi_i}$ has order $k - p$ if $f(\xi)$ has order k and ξ_i has order p .

Definition 2.1: (Nilpotent approximation) The nilpotent approximation along Γ is the germ along Γ of the subriemannian metric we get by truncating a normal orthonormal frame (i.e. a frame from theorem 2.2) at order -1.

Examples:

1) $(n=4, p=2)$ Car with a trailer:

$$\begin{cases} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{y} &= \frac{1}{2}(x_2 u_1 - x_1 u_2) \\ \dot{w} &= \frac{1}{2}\xi(w)x_2(x_2 u_1 - x_1 u_2) \end{cases} \quad (8)$$

with the canonical choice of the surface S explained above. Note that, up to the reparametrization $d\tau = \frac{dw}{\xi(w)}$ of Γ , there is not any parameter in this normal form of nilpotent approximation.

2) $(n=3, \text{generic})$ Unicycle:

$$\begin{cases} \dot{x} &= u \\ \dot{y} &= v \\ \dot{w} &= \gamma(0, 0, w) \frac{1}{2}(yu - xv) \end{cases} \quad (9)$$

and we assume that there is no Martinet point along Γ , i.e. $\gamma(0, 0, w) \neq 0$

In our paper [4], we have shown the following:

Theorem 2.3: (Equivalence to nilpotent approximation)

In all cases under consideration (one-step-bracket-generating or Engel), metric complexity and entropy are equivalent to those of the nilpotent approximation along Γ .

III. CORANK 1 CASE COMPLETE GENERIC CLASSIFICATION

Consider a 1-form ω which is 1 on Γ and vanishing on Δ .

Then, $\alpha = d\omega|_{\Delta}$ is a bilinear form over Δ , that defines a field $A(t)$ of skew-symmetric (w.r.t the subriemannian metric) operators along $\Gamma: [0, T] \rightarrow \mathbb{R}^n$, by:

$$\alpha_t(X, Y) = g(A(t).X, Y)$$

The matrix $A(t)$ does not depend on the choice of ω . Set $\chi(t) = \|A(t)\|_g$. Generically, $\chi(t)$ is smooth. Set $\delta(t) = \left| \frac{\partial \chi}{\partial t} \right|$.

We have the following result ([7], [8], [4]).

Theorem 3.1:

1) Generically, except in dimension 3, the metric-complexity is given by the following formula:

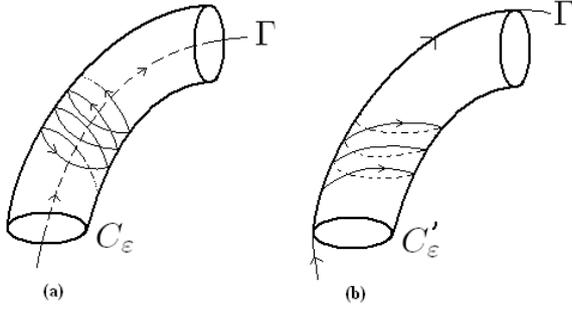


Fig. 1. Contact case: (a) Metric-complexity (b) Entropy

$$MC(\varepsilon) = \frac{2}{\varepsilon^2} \int_{\Gamma} \frac{dt}{\chi(t)} \quad (10)$$

- 2) In dimension 3, either $\chi(t)$ never vanishes, in that case $MC(\varepsilon)$ is still given by formula 10. If $\chi(t)$ vanishes at isolated points t_i , then:

$$MC(\varepsilon) = \frac{-4 \ln(\varepsilon)}{\varepsilon^2} \sum_i \delta(t_i) \quad (11)$$

- 3) The entropy and the metric-complexity are related by the relation:

$$E(\varepsilon) = 2\pi MC(\varepsilon) \quad (12)$$

The point 2) says that *all the time is spent to cross the Martinet surface*, if any.

The point 3) is proven in [4], except in Martinet case. The proof in that case is obtained by using the asymptotic optimal synthesis exhibited below. To show that it is optimal involves the same reasoning as in [8].

Asymptotic optimal syntheses:

1) dimension 3 : contact case

C_ε is the subriemannian cylinder (Fig. 1.a) of radius ε along Γ . Then, the optimal curves for the metric-complexity are just the integral curves of the vector field X_ε , obtained by intersecting the distribution with the tangent planes TC_ε .

There is another *cylinder* (Fig. 1.b), containing the curve Γ . This cylinder is defined in normal coordinates as the cylinder of radius $\frac{\varepsilon}{2}$, centered along a curve Γ' , parallel to Γ , at subriemannian distance $\frac{\varepsilon}{2}$ of Γ . Then, the optimal curves for entropy are the integral curves of the vector field X'_ε , intersection of the distribution with the tangent planes to C'_ε .

In the Martinet case, there is a limit cycle for the vector field X_ε . The asymptotic optimal synthesis for metric-complexity is like that: follow the flow of $-X_\varepsilon$ as long as the vertical coordinate along the cylinder C_ε increases (Fig. 2.a) when it starts to decrease (Fig. 2.b), cross the cylinder by an horizontal subriemannian geodesic realizing the distance

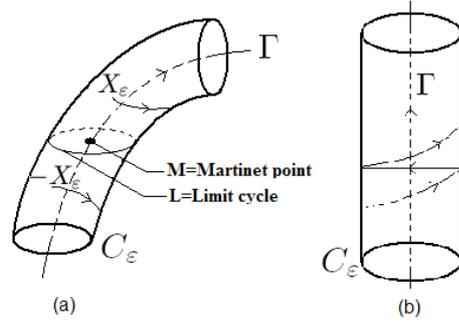


Fig. 2. Martinet Case: Metric-complexity

between C_ε and Γ . After reaching M, take the same strategy reversing time along X_ε .

For entropy, we give some less explicit construction:

- a) Follow the same optimal synthesis as in the contact case (Fig. 1.b) out of the neighborhood of the Martinet point M of height $2k\varepsilon$, for a certain k large enough.
- b) To cross this cylinder of height $2k\varepsilon$, use some strategy of order $\frac{1}{\varepsilon}$ (they are several)

This cost of order $\frac{1}{\varepsilon}$ of step (b) is neglectible in front of the cost of step (a): $\frac{\ln(\varepsilon)}{\varepsilon}$. At the end, it is easily computed that this strategy costs $\frac{-4\pi}{\varepsilon} \ln(\varepsilon) \delta(M)$.

2) Higher dimensional corank 1 cases:

There is no other generic case than contact or quasi-contact.

In these cases, the asymptotic optimal syntheses reduce to the 3 dimensional case in the following way:

In normal coordinates, they are two distinguished x -coordinates corresponding to the maximum modulus eigenvalues of the skew symmetric matrix $A(t)$. Put all the other coordinates to zero, to get a problem in dimension 3.

This strategy, for generic problems, can be realized globally along Γ , since, generically, moduli of eigenvalues of $A(t)$ do not cross each other.

Then, the asymptotic optimal syntheses, for metric-complexity or entropy, are those of the 3 dimensional case, for the restricted problems.

Remark: note that, at this step, up to smooth reparametrization of Γ , whatever the distribution and the metric, these pictures are all globally diffeomorphic in normal coordinates. This is the strong robustness property that we pointed out in the introduction.

It will be the same for all the cases under consideration in the paper, up to corank 3 (even for the Engel's case, which is extremely surprising).

This remark, in our opinion, certainly shows that these control strategies are *very pertinent in practice*.

IV. CORANK 2 SITUATION

In corank 2, there is the exceptional Engel's case (the car with a trailer, example Δ_2 in Introduction), which is generic in dimension 4, and the one-step-bracket-generating cases. There is no other generic situation.

A. One-step-bracket-generating case

In the corank $p > 1$, the 1-forms ω that vanish on the distribution and that are one on $\dot{\Gamma}$ form an affine space of dimension $p - 1$.

Then, taking as in section (II), the forms $\alpha = d\omega|_{\Delta}$, and setting again $g(A_t X, Y) = \alpha(X, Y)$, defines a field A_t along Γ of affine spaces of skew symmetric (w.r.t. g) endomorphisms $\mathcal{A}(t)$ of $\Delta_{\Gamma(t)}$.

The main invariant in this situation is:

$$\chi(t) = \inf_{A_t \in \mathcal{A}(t)} \|A_t\| \quad (13)$$

Of course, $\chi(t)$ coincides, in corank 1, with the invariant defined in section (II).

At this step, another main object comes inside the picture. Consider the mapping $[\cdot, \cdot]_{/\Delta}$ from $\Delta_{\Gamma(t)} \times \Delta_{\Gamma(t)}$ to $T_{\Gamma(t)}\mathbb{R}^n/\Delta_{\Gamma(t)}$ which to two vectors (not vector fields!) (X, Y) associates their bracket $[X, Y]$ modulo Δ . This is a well defined mapping.

Consider B_t , the image of the product of two unit balls U_t :

$$\begin{cases} B_t &= [U_t, U_t]_{/\Delta_{\Gamma(t)}} \\ U_t &= \{X \in \Delta_{\Gamma(t)} \mid \|X\|_g \leq 1\} \end{cases} \quad (14)$$

We say that B_t is convex in the direction of a vector Z if there is a vector $\tilde{Z} = \lambda Z \in B_t$, $\lambda > 0$, and a hyperplane $\{x \mid \omega(x) = 0\}$, such that $\omega(\tilde{Z}) - \omega(x) \geq 0$ for all $x \in B_t$ (i.e., there is a point \tilde{Z} of B_t in the direction of Z such that B_t is entirely on one side of a certain hyperplane through \tilde{Z}).

The main point, which starts to fail to be true in corank $p \geq 3$ is the following.

Theorem 4.1: ([4], [9], [10] for corank $p=1, 2, 3$ generically)

B_t is always convex in the direction of $\dot{\Gamma}_t$

Because of this fact, we have:

Theorem 4.2: ([4], [9], [10])

As soon as (any corank p) B_t is convex in the direction of $\dot{\Gamma}(t)$,

- again, $MC(\Sigma, \varepsilon) = \frac{2}{\varepsilon^2} \int \frac{dt}{\chi(t)}$
- again, $E(\Sigma, \varepsilon) = 2\pi MC(\Sigma, \varepsilon)$

Due to the convexity property, the asymptotic optimal synthesis reduces to the 3-dimensionnal case, by the following procedure:

Take any parametrized surface S , transversal to Δ , and containing Γ . Take a normal coordinate system w.r.t. S , as defined in section (I), and the associated normal form.

- 1) take $A^*(t) \in \mathcal{A}(t)$ that minimizes $\|A(t)\|$, and the 2-dimensional real coordinates x_1, x_2 , corresponding to the eigenspace of A^* , w.r.t. its maximum modulus eigenvalue.
- 2) Put the other x coordinates to zero
- 3) Forget about the y coordinates
- 4) Then, one gets a 3-dimensional contact problem. The asymptotic optimal syntheses are obtained by the 3-dimensional strategies applied to these problems.

B. Engel's case

This is the case of the car with a trailer, distribution Δ_2 of Introduction.

In that case, there is a distinguished choice of the surface S , to define the normal coordinates (explained in section (II)).

There is an abnormal vector field H , contained in the distribution Δ , of length 1, infinitesimal generator of arclength parametrized abnormal trajectories. This vector field has a (length 1) orthogonal I in the distribution.

Set $K = [I, H]$.

$\{I, H, K\}$ defines an orthonormal frame for a corank 1 subriemannian metric (Δ', g') . Generically, g' is not tangent to Γ except at isolated points that do not change anything to the final result.

Let $\chi(t)$ be the invariant relative to the corank one problem defined by Δ', g' and Γ .

Recall that the distinguished surface S is just defined by $S(\theta, t) = \exp(\theta.K(\Gamma(t)))$.

We have the following result:

Theorem 4.3: There is a universal constant $\gamma \simeq 0.00580305$ (called the Berret-Gauthier-Zakalyukin constant), such that:

$$E(\Sigma, \varepsilon) = \frac{3}{2\gamma\varepsilon^3} \int_{\Gamma} \frac{dt}{\chi(t)}$$

Remark: We don't know anything about the metric complexity. In particular, probably, the equality $E = 2\pi MC(\Sigma, \varepsilon)$ fails to be true.

The asymptotic optimal synthesis for the minimum entropy is extremely interesting:

In normal coordinates (x_1, x_2, y, w) relative to the canonical surface S , consider the ε -curves for entropy. The subriemannian metric is $u^2 + v^2$ in the notations of Δ_2 in Introduction.

The optimal curves $(x_1(t), x_2(t))$ are just the Euler's periodic inflexional Elastica.

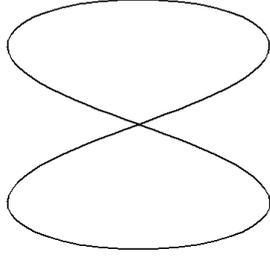


Fig. 3. The dance of minimum entropy

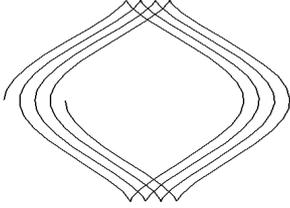


Fig. 4. Trajectory of the center of the wheels of the car

The ε -optimal controls are given in term of the Jacobi elliptic functions by (see [4] and [11])

$$\begin{cases} u(t) &= 1 - dn(K(1 + \frac{4t}{\varepsilon}))^2 \\ v(t) &= -2dn(K(1 + \frac{4t}{\varepsilon}))sn(K(1 + \frac{4t}{\varepsilon}))\sin\frac{\varphi_0}{2} \end{cases}$$

where: $2Eam(K) = K$, and $K(k)$ is the quarter period of the Jacobi elliptic functions of modulus k , $k = \sin\frac{\varphi_0}{2}$ (i.e. $\varphi_0 \approx 130, 692^\circ$)

We have considered, as an application, the less natural non-admissible curve Γ , for the car with a trailer: ($\dot{x} = \cos(\theta)v$, $\dot{y} = \sin(\theta)v$, $\dot{\theta} = u$, $\dot{\varphi} = u - \sin(\varphi)v$)

We want the car to move along a line (the x axis, from point $x=1$ to point $x=0$), but the car and the trailer both remaining perpendicular to this line, i.e.:

$$\Gamma(t) = (1-t, 0, \frac{\pi}{2}, 0)$$

with the notations of section (I).

In normal coordinates, the motion follows the elastica in the (x_1, x_2) plane (Fig. 3).

Trajectories of the center of the wheels of the car are, in the plane (x, y) of natural coordinates, presented on figure 4.

A movie of the motion of the car may be seen at the web address:

<http://www.u-bourgogne.fr/monge/e.busvelle/JPG/>

Remark: There is, in the Engel's case, still the unexpected fact, showing robustness, that, whatever the metric, whatever the Engel's distribution, modulo reparametrization of Γ by the entropy, all ε -optimal pictures are identical, in normal coordinates.

V. HIGHER CORANK

A. Corank 3: one-step-bracket-generating

In the case of corank 3, the situation is already described by theorems 4.1, 4.2 in the previous section, despite some isolated points of Γ that introduce technical complications (see [4]).

The remaining $p = 3$ generic cases are $n = 2$, $n = 3$ (where a surface of degeneracy appears).

In both cases, were able to compute what happens. We don't state these results here.

B. Corank $p \geq 3$

Let us say a few words about the situation in the one-step-bracket-generating case.

We have the body B_t (defined in the previous section), moving along the curve Γ_t . From corank 6 on, the new fact is that (generically), the body B_t is never convex in the direction of Γ_t .

Then, using deeply the structure of quaternions, it can be shown that (corank 6), entropy verifies:

$$\frac{4\pi}{\varepsilon} \int_{\Gamma} \frac{dt}{\chi(t)} \leq E(\Sigma, \varepsilon) \leq \frac{6\pi}{\varepsilon^2} \int_{\Gamma} \frac{dt}{\chi(t)} \quad (15)$$

The lower bound is reached in the B_t convex case.

The upper bound is reached in the worst case.

For coranks 4, 5, some intermediate situations appear: the body B_t (moving along Γ_t) may be on some open intervals convex in the direction of Γ_t , on some other intervals non convex.

Then, generically, we always have formula (15), the lower bound being eventually attained on open subintervals of Γ .

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