

# Design principles for muscle-like variable impedance actuators with noise rejection property via co-contraction

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**Abstract**—In this paper we suggest basic principles to design a novel type of passive variable impedance actuator aimed at replicating a specific property of human co-contraction, related to the ability to cope with uncertainties affecting any physical/biological system. In particular the dynamical model of the proposed actuator is such that the variance of the state vector in response to noisy disturbances can be reduced by tuning the passive stiffness of the system. By means of a linearization analysis we characterize the mathematical properties that a non-linear dynamical system should have in order to possess this noise rejection property. We provide a practical example of such a system based on non-linear springs whose critical feature is to attach some elastic elements to a fixed reference (e.g. “ground”). We then show that this antagonist actuator structure is actually analog to Hill’s model of the human muscle/tendon system, emphasizing its biological relevance. We finally illustrate how time-varying stiffness can be efficiently planned feedforwardly to reject disturbances that may affect task achievement. To this aim, we use the formalism of stochastic optimal control to derive open-loop controls anticipating the consequences of unpredictability and instability linked to the task. We conclude that the suggested actuators are well-suited to mimic the main features of human co-contraction and plan to implement this type of actuator on the robot platform iCub in a near future.

variable impedance actuator, stiffness, co-contraction, stochastic optimal control, uncertainty

## I. INTRODUCTION

Muscle co-contraction can be modeled as an active modulation of the passive musculo-skeletal compliance. Within this context, some findings in human motor control have shown that active compliance modulation is fundamental when planning movements in presence of unpredictability and uncertainties [1], [2]. Along this line of research, the present article investigates the link between active impedance control and unpredictability, with special focus on robotic applications. A necessary prerequisite for fully understanding the motivations behind the present paper is our previous work [3] where we observed that different actuator models behave differently in handling unpredictability in open-loop movement planning with a two degrees of freedom unstable manipulator. The present paper gives a mathematical characterization of the properties that a system (and its actuators) should have in order to arbitrarily reduce the effects of un-

predictability without relying on active feedback<sup>1</sup>. The characterization is given for a class of manipulators actuated with variable impedance actuators but the analytical computations are restricted to a single joint. System properties are analyzed and compared with Hill’s muscle model. We eventually use tools from stochastic optimal control to illustrate the possibility of efficiently exploiting stiffness control to reject unpredictable disturbances in open-loop.

## II. METHODS

### A. Preliminary

Throughout this article, we will consider a specific class of manipulators with elastic joints, whose dynamics can be written as follows [4, Chap. 13]:

$$\ddot{\mathbf{q}} = \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q} - \boldsymbol{\theta}), \quad (1)$$

$$\ddot{\boldsymbol{\theta}} = \mathbf{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \mathbf{u}) - \mathbf{g}(\mathbf{q} - \boldsymbol{\theta}). \quad (2)$$

The variable  $\mathbf{q}$  denotes generalized coordinates of the external system (manipulator level), while the variable  $\boldsymbol{\theta}$  denotes the generalized coordinates of the internal system (actuator level). Both systems are coupled via the function  $\mathbf{g}$ , which represents an elastic (typically non-linear) element coupling the manipulator with the motor. The functions  $\mathbf{f}$  and  $\mathbf{h}$  thus represent the external (manipulator) and internal (actuator) dynamics, respectively. The control input  $\mathbf{u}$  first acts on the internal system, which in turns activates the external one. The system (1)-(2) can be also used to represent manipulators actuated with variable stiffness actuators. In this specific class of systems, the characteristics of the elastic element  $\mathbf{g}$  can be either changed directly [5] (acting on an additional input, see also Section II.II-B1) or indirectly [6] (by traversing equilibrium states with constant  $\mathbf{q}$  but different  $\boldsymbol{\theta}$ , see Section II.II-C).

The aim of this paper is to understand the properties that (1)-(2) should have in order to guarantee that the

<sup>1</sup>The reader interested in understanding the relevance of (feedback free) open-loop planning should refer to [3] and its introduction.

rejection of disturbances acting on the system can be (arbitrarily) augmented without using active feedback loops. The proposed analysis relies on a linearization of the system dynamics around an equilibrium configuration and therefore the results can be extended to the original non-linear system only locally.

Let us assume the existence of an equilibrium state  $\mathbf{x}_{eq}^\top = (\mathbf{q}_{eq}^\top, \boldsymbol{\theta}_{eq}^\top, \dot{\mathbf{q}}_{eq}^\top, \dot{\boldsymbol{\theta}}_{eq}^\top)$  with  $\mathbf{u} = \mathbf{u}_{eq}$  and let us linearize the system (1)-(2) around that point:

$$\begin{aligned}\ddot{\mathbf{q}} &= \left. \frac{\partial \mathbf{f}}{\partial \mathbf{q}} \right|_{eq} \delta \mathbf{q} + \left. \frac{\partial \mathbf{f}}{\partial \dot{\mathbf{q}}} \right|_{eq} \delta \dot{\mathbf{q}} + \left. \frac{\partial \mathbf{g}}{\partial \mathbf{z}} \right|_{eq} (\delta \mathbf{q} - \delta \boldsymbol{\theta}), \\ \ddot{\boldsymbol{\theta}} &= \left. \frac{\partial \mathbf{h}}{\partial \boldsymbol{\theta}} \right|_{eq} \delta \boldsymbol{\theta} + \left. \frac{\partial \mathbf{h}}{\partial \dot{\boldsymbol{\theta}}} \right|_{eq} \delta \dot{\boldsymbol{\theta}} - \left. \frac{\partial \mathbf{g}}{\partial \mathbf{z}} \right|_{eq} (\delta \mathbf{q} - \delta \boldsymbol{\theta}) + \left. \frac{\partial \mathbf{h}}{\partial \mathbf{u}} \right|_{eq} \delta \mathbf{u}.\end{aligned}$$

In the latter equation,  $\delta$  denotes deviation of a certain variable with respect to the equilibrium configuration and the quantity  $\partial \mathbf{g} / \partial \mathbf{z}$  denotes the derivative of the function  $\mathbf{z} \mapsto \mathbf{g}(\mathbf{z})$ , which will be denoted by  $\mathbf{g}'$ . Therefore, around an equilibrium configuration, we get a linear system of the form:

$$\dot{\delta \mathbf{x}} = A \delta \mathbf{x} + B \delta \mathbf{u},$$

which can be expanded as follows<sup>2</sup>:

$$\begin{aligned}\dot{\delta \mathbf{x}} &= \left( \begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline \frac{\partial \mathbf{f}}{\partial \mathbf{q}} + \mathbf{g}' & -\mathbf{g}' & \frac{\partial \mathbf{f}}{\partial \dot{\mathbf{q}}} & 0 \\ -\mathbf{g}' & \frac{\partial \mathbf{h}}{\partial \boldsymbol{\theta}} + \mathbf{g}' & 0 & \frac{\partial \mathbf{h}}{\partial \dot{\boldsymbol{\theta}}} \end{array} \right)_{eq} \delta \mathbf{x} \\ &+ \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ \frac{\partial \mathbf{h}}{\partial \mathbf{u}} \end{array} \right)_{eq} \delta \mathbf{u}.\end{aligned}$$

### B. Stability analysis

As previously pointed out, we are interested in understanding what properties (1)-(2) should have in order to vary the system noise rejection capabilities without active feedback loops. In this section, the analysis is conducted on the linearized system and therefore results will hold only locally in the original non-linear dynamics. In order to model uncertainty, let us define an input matrix  $C$  through which noise, denoted by  $\mathbf{w}$ , affects the system dynamics:

$$\dot{\delta \mathbf{x}} = A \delta \mathbf{x} + B \delta \mathbf{u} + C \mathbf{w},$$

where  $\mathbf{w}$  is a standard Gaussian process (i.e. normalized white noise). In order to characterize the system capability of rejecting the disturbances due to  $\mathbf{w}$ , we will

<sup>2</sup>Note that the matrix  $A$  is block companion, which might be exploited in the study of its stability. However, in this paper, we will directly assume that  $A$  is asymptotically stable.

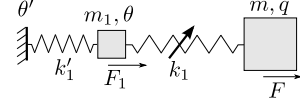


Figure 1. Simple idealized variable stiffness actuator, with  $\frac{\partial \mathbf{h}}{\partial \boldsymbol{\theta}}|_{eq} \neq 0$ . The possibility to directly vary the stiffness  $k_1$  is emphasized by the oblique arrow over the spring. In this example we thus have  $\mathbf{g}'|_{eq} = -k_1$ . For simplicity only, we will choose unitary masses (i.e.  $m = m_1 = 1$ , see (3)).

focus on the properties of the matrix  $P$  defined to be the asymptotic variance of the state  $\delta \mathbf{x}$ , i.e.:

$$P = \lim_{t \rightarrow \infty} \text{cov}[\delta \mathbf{x}(t)],$$

when no changes to the input are applied (i.e.  $\delta \mathbf{u} = \mathbf{0}$ ). Necessary condition for  $P$  to be finite is the stability of the matrix  $A$  [7]. Under this assumption, the matrix  $P$  corresponds to the unique solution of the following continuous Lyapunov equation:

$$AP + PA^\top + CC^\top = 0.$$

In the following analysis we will try to characterize the properties of the matrix  $P$  for a class of variable impedance actuators. In the remaining of the paper, in order to make the analytical computations tractable, we restrict ourselves to the case  $\mathbf{q} = q \in \mathbb{R}$ . In practice, we thus limit our analysis to systems with only one external joint. Even if this might appear a major restriction, we believe that this is an admissible simplification in order to keep the analytical computations compact while focusing on an essential building block, i.e. the single joint actuator. Another evident simplification will be the restriction to linear (vs rotational) systems. In this case however we do not lose generality because identical considerations will hold for the analogous rotational systems.

*1) Stability analysis for manipulators with variable impedance joints in serial configuration:* Manipulators with elastic joints [8] do not have the ability to change the passive properties of the elastic element  $\mathbf{g}'$ . A class of variable impedance actuators (see for example [5], [9]) can instead vary the value of  $\mathbf{g}'$  thus varying the matrix  $P$ . In these actuators a variable stiffness is placed in between the motor and the actuated joint (thus the name “serial configuration”). A linear equivalent of a single joint actuated with this kind of actuator is shown in Fig. 1. The dynamic equation of this system is characterized by the fact that they can be written in the form (1)-(2) with  $\mathbf{q} = q \in \mathbb{R}$  and  $\boldsymbol{\theta} = \theta \in \mathbb{R}$ , respectively the external and internal states. Following the notation in Figure 1 we obtain:

$$\begin{aligned}\ddot{q} &= -k_1(q - \theta) - b_0\dot{q} + F, \\ \ddot{\theta} &= k_1(q - \theta) + k_1'(\theta' - \theta) - b_1\dot{\theta} + F_1.\end{aligned}\quad (3)$$

where  $k_1$ ,  $k_1'$  and  $b_0$ ,  $b_1$  are respectively spring's stiffness and friction constants and  $F$  and  $F_1$  are external forces applied to the masses.

From the above equation, we identify:

$$\mathbf{g}' = -k_1, \quad \frac{\partial \mathbf{f}}{\partial \mathbf{q}} = 0, \quad \frac{\partial \mathbf{h}}{\partial \boldsymbol{\theta}} = -k'_1, \quad \frac{\partial \mathbf{f}}{\partial \dot{\mathbf{q}}} = -b_0, \quad \frac{\partial \mathbf{h}}{\partial \dot{\boldsymbol{\theta}}} = -b_1.$$

To simplify our notations but keep things general, we rewrite the matrix  $A$  as follows:

$$A = \left( \begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -a-c & c & -d & 0 \\ c & -b-c & 0 & -e \end{array} \right).$$

All the above quantities are scalars in the one-dimensional case, and their physical meaning can be easily understood from Figure 1 together with (3). Note that in the latter example we have  $a = 0$ ,  $b = k'_1$  and  $c = k_1$  (the stiffness we can actually vary), and the terms  $d$  and  $e$  are friction coefficients (note that all quantities are positive).

Let us assume that noise enters into the system via a term  $C = (0, 0, \sigma_1, \sigma_2)^\top$ , such that noise affects both the internal and external dynamics. We are interested in understanding the properties of the Lyapunov solution  $P$  when varying  $c$  which corresponds to the variable elastic element. The lower the eigenvalues of  $P$ , the bigger the noise rejection of the system. The first observation is that  $P$  is finite only if  $A$  is stable and therefore we need to assume that  $a$  and  $b$  cannot be simultaneously zero<sup>3</sup>. This implies that there must exist either a spring connecting  $q$  to the ground or a spring connecting  $\theta$  to the ground ( $k'_1$  in Fig. 1). Actuators like the ones proposed in [5], [9] do not typically have these springs since noise rejection is achieved with active feedback loops.

Let us now characterize the behavior of  $P$  under variation of  $c$ . Explicit calculations with Maple give the following formulae:

$$\begin{aligned} \lim_{c \rightarrow 0} \text{var}(q) &= \frac{1}{2} \frac{\sigma_1^2}{ad}; \quad \lim_{c \rightarrow \infty} \text{var}(q) = \frac{1}{2} \frac{(\sigma_1 + \sigma_2)^2}{(a+b)(d+e)}, \\ \lim_{c \rightarrow 0} \text{var}(\dot{q}) &= \frac{1}{2} \frac{\sigma_1^2}{d}; \quad \lim_{c \rightarrow \infty} \text{var}(\dot{q}) = \frac{1}{2} \frac{\sigma_1^2 + \sigma_2^2}{d+e}, \\ \lim_{c \rightarrow 0} \text{var}(\theta) &= \frac{1}{2} \frac{\sigma_2^2}{be}; \quad \lim_{c \rightarrow \infty} \text{var}(\theta) = \frac{1}{2} \frac{(\sigma_1 + \sigma_2)^2}{(a+b)(d+e)}, \\ \lim_{c \rightarrow 0} \text{var}(\dot{\theta}) &= \frac{1}{2} \frac{\sigma_2^2}{e}; \quad \lim_{c \rightarrow \infty} \text{var}(\dot{\theta}) = \frac{1}{2} \frac{\sigma_1^2 + \sigma_2^2}{d+e}. \end{aligned}$$

Regarding our example, it is clear that the variance of the external state  $q$  can be decreased by making  $b$  and/or  $e$  larger. Increasing  $b$  just means choosing a stiffer spring that connects the internal state to the ground (i.e.  $k'_1$ ), while increasing  $e$  would imply having larger viscous friction forces acting on  $\theta$ . Similarly, for the trace of  $P$ ,

<sup>3</sup>If  $a = b = 0$  the first two columns of  $A$  become linearly dependent. In such a case, zero is an eigenvalue of  $A$  and thus the system can be at best marginally stable.

which is a measure of the overall system sensitivity to noise<sup>4</sup>, we get the following limits:

$$\begin{aligned} \lim_{c \rightarrow 0} \text{trace}(P) &= \frac{1}{2} \frac{\sigma_1^2 be(a+1) + \sigma_2^2 ad(b+1)}{abde}, \\ \lim_{c \rightarrow \infty} \text{trace}(P) &= \frac{(a+b)(\sigma_1^2 + \sigma_2^2) + (\sigma_1 + \sigma_2)^2}{(a+b)(d+e)}. \end{aligned}$$

If we further assume that  $e = d = 1$ , the full expression for  $\text{trace}(P)$  and its partial derivative with respect to  $c$  reduce to:

$$\begin{aligned} \text{trace}(P) &= \frac{1}{2} \frac{((\sigma_1 + \sigma_2)^2 + (\sigma_1^2 + \sigma_2^2)(a+b))c}{2 \frac{ac+bc+ab}{b(a+1)\sigma_1^2 + a(b+1)\sigma_2^2} + \frac{1}{2} \frac{ac+bc+ab}{(a\sigma_2 - b\sigma_1)^2}}, \\ \frac{\partial \text{trace}(P)}{\partial c} &= -\frac{1}{2} \frac{(a\sigma_2 - b\sigma_1)^2}{(ac+bc+ab)^2} < 0. \end{aligned}$$

The above formulae show that this kind of system possesses the interesting property of monotonically diminishing the effect of noise by increasing  $c$ , i.e. the stiffness of the spring connecting the actuator side  $\theta$  to the joint side  $q$ . However, this noise rejection capability crucially depends on  $a$  and  $b$  (stiffness of the springs connecting the actuator and the joint to the ground) and on  $d$  and  $e$  (viscous friction on the actuator and on the joint) which determine a lower bound for the noise rejection via the equations above.

### C. Practical example with antagonist non-linear springs

In this subsection, we provide an example of an antagonist system possessing the above mentioned property. Let us consider a system of 3 masses and 4 springs, as depicted in Figure 2.

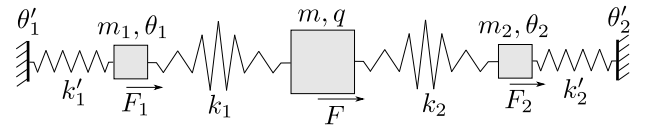


Figure 2. Example of physical system with passive variable stiffness. The springs with stiffnesses  $k_1$  and  $k_2$  are non-linear (here cubic). The two other springs are assumed to be linear here, but they could be non-linear as well. It is noted that this system can be shown to be formally equivalent to the one presented in Fig. 1.

Using Lagrangian mechanics and assuming cubic non-linear springs where indicated, we can write the dynamics as follows:

$$\begin{aligned} m\ddot{q} &= k_1(\theta_1 - q)^3 + k_2(\theta_2 - q)^3 - b_0\dot{q} + F, \\ m_1\ddot{\theta}_1 &= k_1(q - \theta_1)^3 + k'_1(\theta'_1 - \theta_1) - b_1\dot{\theta}_1 + F_1, \\ m_2\ddot{\theta}_2 &= k_2(q - \theta_2)^3 + k'_2(\theta'_2 - \theta_2) - b_2\dot{\theta}_2 + F_2. \end{aligned} \quad (4)$$

Remarkably, the dynamic equation of this system can be written in the form (1)-(2) with  $\mathbf{q} = q \in \mathbb{R}$  and

<sup>4</sup>We are using here the property that the trace of a matrix is the sum of its eigenvalues.

$\theta = (\theta_1, \theta_2)^\top \in \mathbb{R}^2$ . Let us consider an equilibrium configuration and the linearized system around it:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{3k_1}{m}(\theta_1 - q)^2 - \frac{3k_2}{m}(\theta_2 - q)^2 & \frac{3k_1}{m}(q - \theta_1)^2 & -\frac{3k_1}{m_1}(q - \theta_1)^2 - \frac{k'_1}{m_1} & 0 \\ \frac{3k_1}{m_1}(\theta_1 - q)^2 & -\frac{3k_1}{m_1}(q - \theta_1)^2 - \frac{k'_1}{m_1} & 0 & 0 \\ \frac{3k_2}{m_2}(\theta_2 - q)^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{3k_2}{m}(q - \theta_2)^2 & -\frac{b_0}{m} & 0 & 0 \\ 0 & 0 & -\frac{b_1}{m_1} & 0 \\ -\frac{k'_2}{m_2} - \frac{3k_2}{m_2}(q - \theta_2)^2 & 0 & 0 & -\frac{b_2}{m_2} \end{pmatrix}_{|eq},$$

and

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1/m_2 \\ 0 & 0 & 0 & 0 & 1/m_1 & 0 \end{pmatrix}^\top.$$

Then the linearized system writes:

$$\dot{\delta \mathbf{x}} = A\delta \mathbf{x} + B\delta \mathbf{u},$$

with  $\mathbf{x} = (q, \theta_1, \theta_2, \dot{q}, \dot{\theta}_1, \dot{\theta}_2)^\top$ ,  $\mathbf{u} = (F_1, F_2)^\top$ ,  $\delta \mathbf{x} = \mathbf{x} - \mathbf{x}_{eq}$  and  $\delta \mathbf{u} = \mathbf{u} - \mathbf{u}_{eq}$ . Also in this case the linearized system is only marginally stable if we remove both the springs  $k'_1$  and  $k'_2$  connecting the actuators to the ground. The noise rejection characteristic of the system can be increased by traversing states with constant equilibrium for the variable  $q$  but different equilibria for the variables  $\theta_1, \theta_2$ . This allows to vary the value of  $\mathbf{g}'(\mathbf{x}_{eq})$  via changes of  $\mathbf{x}_{eq}$  which do not affect  $q_{eq}$ .

For instance for the system (4) the stiffness “felt” at the middle mass, locally around an equilibrium point  $\mathbf{x}_{eq}$ , is expressed as:

$$K_q = \frac{3}{m} (k_1(\theta_{1,eq} - q_{eq})^2 + k_2(\theta_{2,eq} - q_{eq})^2).$$

From that equation it is obvious to see that stretching the internal springs would cause an increase of the stiffness  $K_q$ . Interestingly new static equilibria with larger stiffness may be achieved without changing  $q_{eq}$ . For instance if  $q_{eq}$  is given and fixed, then for any  $\theta_{2,eq}$  the variable  $\theta_{1,eq}$  can be chosen as follows:

$$\theta_{1,eq} = q_{eq} + \left( \frac{1}{k_1} (-k_2(\theta_{2,eq} - q_{eq})^3 - F) \right)^{1/3}.$$

Then, we can simply define the control vector  $\mathbf{u}_{eq} = (F_{1,eq}, F_{2,eq})^\top$  that will realize the equilibrium configuration  $\mathbf{x}_{eq}$  as:

$$\begin{aligned} F_{1,eq} &= -k_1(q_{eq} - \theta_{1,eq})^3 - k'_1(\theta'_1 - \theta_{1,eq}), \\ F_{2,eq} &= -k_2(q_{eq} - \theta_{2,eq})^3 - k'_2(\theta'_2 - \theta_{2,eq}). \end{aligned}$$

In the following, we illustrate the ability for this system to decrease  $\text{trace}(P)$  by increasing  $\theta_{2,eq}$  (with  $\theta_{1,eq} =$

$-\theta_{2,eq}$ ) (Fig. 3). The asymptotic value of  $\text{trace}(P)$  (its lower bound) can be decreased by increasing the values of  $k'_1$  and  $k'_2$  (solid line vs dashed-line).

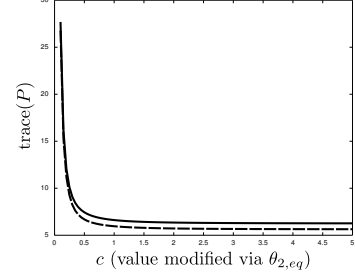


Figure 3. Trace of the Lyapunov matrix corresponding to the linearization of the system (4). In abscisse, the position of  $\theta_{2,eq}$  is depicted. It is implicitly assumed that the configuration is kept symmetric, i.e.  $\theta_{1,eq} = -\theta_{2,eq}$  and  $q_{eq} = 0$ . Numerical values used are reported in Section III. The dashed line illustrates the decrease of the asymptotic value of  $\text{trace}(P)$  when increasing the values of  $k'_1$  and  $k'_2$  by a factor 2.

#### D. Analogy with Hill's muscle model

In this subsection, we emphasize that the actuator proposed in the previous section has interesting analogies with the human musculoskeletal system. In particular, we compare our actuator to Hill's model of muscles. Hill's antagonist muscles model is illustrated in the Figure 4.

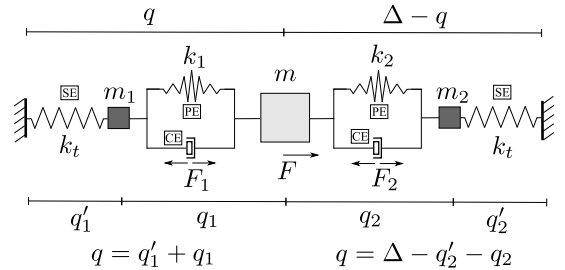


Figure 4. Hill's antagonist muscles model with non-linear springs. SE=series elastic element, PE=parallel elastic element, CE=contractile element. The total length of the structure is fixed, equal to  $\Delta$ . Usually Hill's model assumes linear springs but for the purpose of this paper we consider non-linear ones (which is more realistic anyway).

Again, we make use of the Lagrangian formalism to derive the equation. We first write the potential and kinetic energy functions:

$$\begin{aligned} E_p &= \frac{1}{2}k_t q_1'^2 + \frac{1}{4}k_1 q_1^4 + \frac{1}{2}k_t q_2'^2 + \frac{1}{4}k_2 q_2^4, \\ E_k &= \frac{1}{2}m\dot{q}^2 + \frac{1}{2}m_1(\dot{q} - \dot{q}_1)^2 + \frac{1}{2}m_2(\dot{q} + \dot{q}_2)^2. \end{aligned}$$

Applying Euler-Lagrange equations and noting that  $q'_1 = q - q_1$  and  $q'_2 = \Delta - q - q_2$  gives:

$$\begin{aligned} F_1 &= -k_t(q - q_1) + k_1 q_1^3 - m_1(\ddot{q} - \ddot{q}_1), \\ F_2 &= -k_t(\Delta - q - q_2) + k_2 q_2^3 + m_2(\ddot{q} + \ddot{q}_2), \\ F &= k_t(q - q_1) - k_t(\Delta - q - q_2) + m\ddot{q} \\ &\quad + m_2(\ddot{q} + \ddot{q}_2) + m_1(\ddot{q} - \ddot{q}_1). \end{aligned}$$

Then inserting the two first rows into the last one and simplifying the terms involving  $k_t$ , we get:

$$F = m\ddot{q} + F_2 - k_2 q_2^3 + k_1 q_1^3 - F_1.$$

Using the change of variables  $\theta_1 = q'_1 = q - q_1$  and  $\theta_2 = \Delta - q'_2 = q + q_2$  we obtain:

$$q_1 = q - \theta_1 \text{ and } q_2 = \theta_2 - q.$$

Therefore, the dynamical system corresponding to Hill's model can be re-written as:

$$\begin{aligned} m\ddot{q} &= -k_1(q - \theta_1)^3 - k_2(q - \theta_2)^3 + F + F_1 - F_2, \\ m_1\ddot{\theta}_1 &= -k_t\theta_1 - k_1(\theta_1 - q)^3 + F_1, \\ m_2\ddot{\theta}_2 &= -k_t(\Delta - \theta_2) - k_2(\theta_2 - q)^3 + F_2. \end{aligned}$$

This system has to be compared with the one proposed previously in (4), with  $\theta'_1 = 0$  and  $\theta'_2 = \Delta$ .

It is then obvious that the additional springs added on the extreme left and right sides of the system (Fig. 2) and that connect to the ground simply play the role of tendons (elastic elements attached to the bones, which serve as a fixed reference). The only difference is the way the generalized forces enter into the system: this is because our system directly actuates the masses  $m_1$  and  $m_2$ , while forces are generated through contractile elements in Hill's muscle model. These considerations show that the class of actuators we presently describe is compatible with the most classical model of the human tendon/muscle system.

### III. SIMULATIONS

In this section, we consider a simple unstable task to illustrate how stiffness control as offered by the proposed actuator can be used effectively. We consider the dynamical system given in (4) and the goal of the task is to maintain the middle mass at position zero without feedback but by tuning the stiffness adequately. The presence of uncertainty (i.e. dynamical noise) and instability (divergent force field) makes the task difficult if one considers latencies in the feedback loop or simply neglects feedbacks for instance.

To find an effective solution, we build a stochastic optimal control (SOC) problem, and we used an algorithm for SOC using (Feynman's) path integrals, similar to PI<sup>2</sup> as recently developed by [10]. Our algorithm jointly relies on the theories developed by [10], [11], but the precise description of the algorithm we used is out of the scope of the present paper. The method allows to automatically derive an open-loop SOC that optimally regulates the system in a divergent force field centred at  $q = 0$  (see [1] for a similar unstable task studied in humans).

In state space form, using  $\mathbf{x} = (q, \theta_1, \theta_2, \dot{q}, \dot{\theta}_1, \dot{\theta}_2)^\top$  and  $\mathbf{u} = (F_1, F_2)^\top$  we can rewrite the system (4) in the control-affine form:

$$d\mathbf{x} = \mathbf{a}(\mathbf{x})dt + \mathbf{B}u dt + \mathbf{C}d\mathbf{w},$$

where  $\mathbf{w}$  is a standard 2-D Brownian motion.

To define a task and attribute a meaning to "optimal", we consider a cost function under the general form:

$$J(t_0, \mathbf{x}_0) = \mathbb{E}\left(\phi(\mathbf{x}(t_f))\right) + \int_{t_0}^{t_f} q(\mathbf{x}) + \mathbf{u}^\top R \mathbf{u} dt.$$

Let us further assume, for technical reasons, that  $C = B\sqrt{\lambda R^{-1}}$  for an arbitrary  $\lambda$ . For this class of SOC problems, it has been shown that the solutions can be estimated using path integral approximations [10], [11].

Here we used  $\phi(\mathbf{x}) = q(\mathbf{x}) = \mathbf{x}^\top Q \mathbf{x}$ , with  $Q = \text{diag}(0, 0, 1e^4, 1e^1, 1e^1, 1e^2)$ ,  $R = \text{Id}_{2 \times 2}$ ,  $\lambda = 0.1$ . The initial and final configurations were defined as  $\mathbf{x}(t_f) = \mathbf{0}$  and  $\mathbf{x}(0) = (0, -1.5, 1.5, 0, 0, 0)^\top$  and the task duration  $t_f$  was set to 10 seconds. The springs characteristics were:  $m = m_i = 0.1$ ,  $k_i = k'_i = 2$  and  $b_0 = b_i = 2$ ,  $i = 1, 2$ . The ground position was  $\theta'_1 = \theta'_2 = 0$ . Note that we also added a cost to penalize unrealistic behaviors, such as having a spring pass through the ground. More precisely, we added  $\frac{1}{2}10^{10}(\tanh(10^5\theta_1 + 1) - \tanh(10^5\theta_2 - 1))$  to the term  $q(\mathbf{x})$ .

For the simulations we chose a step size  $dt = 0.02$  s. The algorithm relies on an iterative procedure, using importance sampling of paths ( $N = 1000$  samples at each update), for a total of 250 iterations. To evaluate the effect of varying instability of the task, we used two divergent force fields<sup>5</sup> of different magnitudes ( $F = K_{\text{div}}q$  with  $K_{\text{div}} = 2$  or  $6$  N/m). The goal of the task was to maintain the middle mass around position  $q = 0$  for 10 seconds using an open-loop control and minimizing the amount of control effort involved. Note that minimizing the control energy makes perfectly sense here because large stiffness implies large control norm since springs must be stretched significantly. This property is similar to the large metabolic energy expenditure associated with co-contraction in humans, which thus motivates the design of "just stiff enough" control laws.

To initialize the process we used a pre-stretched configuration of the springs which resulted in a quite large stiffness at the beginning, sufficient to counteract the instability of the task but energy consuming. Therefore, this is not optimal since the task is performed but with a high control cost (i.e. large co-contraction). Across iterations, a more efficient open-loop control is uncovered, setting the optimal trade-off between remaining close to  $q = 0$  and using the smallest control energy possible (see Fig. 5).

<sup>5</sup>Although it is divergent, the force field is nevertheless deterministic. It is distinct from the term involving the Brownian motion  $\mathbf{w}$ , which just reflects the noise affecting the system. Combined together, instability (force field) and uncertainty (noise) makes it impossible to stabilize the system at  $q = 0$  in open-loop without tuning the mechanical impedance.

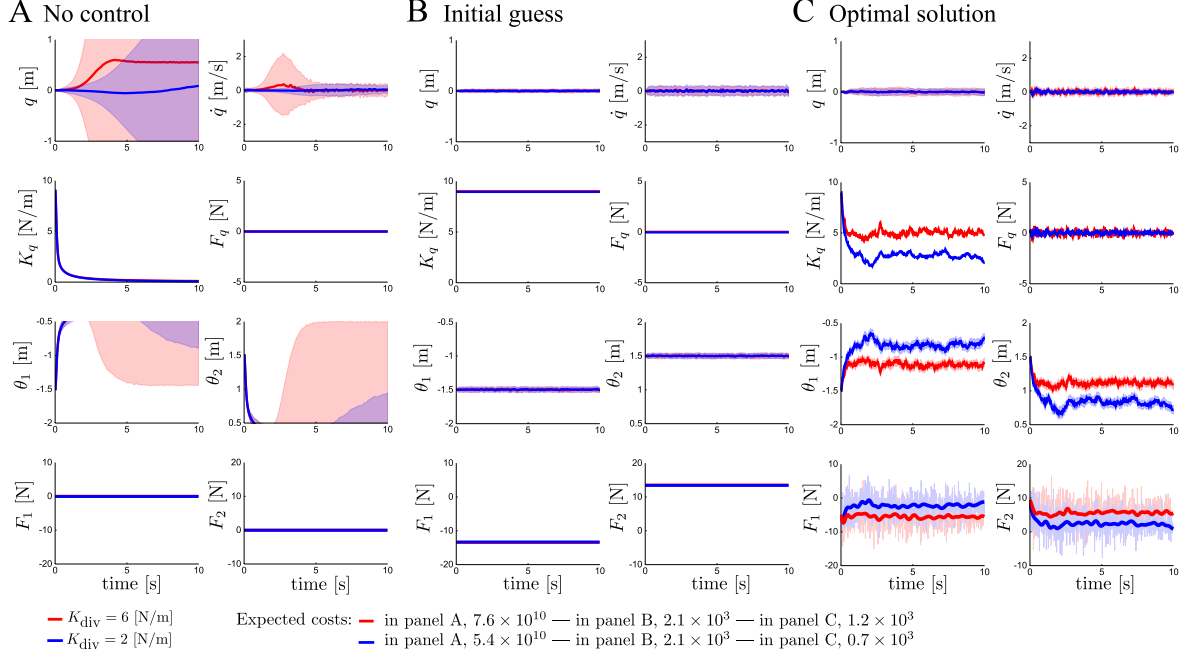


Figure 5. Stochastic optimal control simulations for an unstable task. A. The uncontrolled system diverges very rapidly, even with in quite light divergent force field ( $K_{\text{div}} = 2$  N/m, blue curves). Shaded areas depict the standard deviation across 100 trials. Black thick traces depict the average behavior. B. Using a relatively high stiffness, the task can be performed quite accurately in open-loop, despite uncertainty and instability. C. An optimal trade-off between accurate trajectory tracking and low energy consumption is found by solving a SOC problem. As a result, the springs are stretched optimally to perform the task well enough. No feedback is used in those simulations, meaning that they will still be an effective solution in presence of transmission delays. In the graphs, we have defined  $F_q = k_1(\theta_1 - q)^3 + k_2(\theta_2 - q)^3$  and  $K_q = 3(k_1(\theta_1 - q)^2 + k_2(\theta_2 - q)^2)$ , which are the feedforward force and stiffness active at the endpoint (i.e. the external state  $q$ ). The optimal control is relatively noisy (bottom right plots) due to the stochastic nature of the sampling process. A filtered version of the control forces is also plotted to better visualize the trend in the behavior of the optimal control law.

#### IV. CONCLUSION

We have presented a passive variable stiffness actuator with interesting human-like features. Mainly it makes it possible to mimic some effects of muscle co-contraction in humans, which is useful to cope with noise and sensorimotor delays affecting physical/biological systems. The price to pay to achieve this behavior is a waste of energy associated with the need for stretching additional springs. This property is nevertheless similar to the large energy expenditure of muscle co-contraction and this moreover justifies the use of optimal control techniques to reduce this energy consumption to a minimum. This also justifies why and how stiffness should be tuned in such systems, going beyond safety or force regulation considerations usually investigated in the field. We illustrated the use of variable stiffness during an unstable stabilization task. Open-loop stochastic optimal control laws were derived using an algorithm based on path integrals. Future work will focus on the actual fabrication and implementation of such actuators on a real humanoid platform (iCub) in order to consider more realistic daily life unstable tasks, such as screw driving for instance.

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