A Biomechanical Inactivation Principle

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Abstract—This paper develops the mathematical side of a theory of inactivations in human biomechanics. This theory has been validated by practical experiments, including zero-gravity experiments. The theory mostly relies on Pontryagin's maximum principle on the one side and on transversality theory on the other side. It turns out that the periods of silence in the activation of muscles that are observed in practice during the motions of the arm can appear only if "something like the energy expenditure" is minimized. Conversely, minimization of a criterion taking into account the "energy expenditure" guaranties the presence of these periods of silence, for sufficiently short movements.

1. INTRODUCTION

In order to perform accurate goal-directed movements, the central nervous system (CNS) has to compute neural commands according to the initial state of the body, the location of the target, and the external forces acting on the limbs. Arm movement planning requires solving redundancy problems related to angular displacements, joint torques, muscular patterns, and neural inputs [4].

Experimental studies reported stereotypical kinematic features during pointing and reaching arm movements (e.g., quasi-straight finger paths, bell-shaped finger velocity profiles [1, 22, 30]). These features were found to be robust despite changes in mass, initial/final positions, amplitudes, and speeds of displacements [2, 7, 18, 19, 25].

Therefore, many studies have attempted to identify the principles of motion planning and control, hypothesizing that movements were optimal with respect to some criteria. The present article addresses the question whether motor planning is optimal according to an identifiable criterion.

This question amounts to solving an *inverse optimal control problem*: given recorded experimental data, try to infer a cost function with regard to which the observed behaviour is optimal [31]. In the theory of linear-quadratic control, the question of which quadratic cost is minimized in order to control a linear system along certain trajectories was already raised by R. Kalman [20]. Some methods allowed deducing cost functions from optimal behaviour in system and control theory (linear matrix inequalities [9]) and in Markov decision processes (inverse reinforcement learning [23]).

The present work introduces a new methodological point of view in inverse optimal control problems.

The starting point is the observation of simultaneous inactivation of opposing muscles during movements presumed as optimal. Using mathematical transversality arguments from differential topology, we proved that the minimization of a nonsmooth cost is a necessary condition to obtain these inactivation phases along optimal trajectories.

The second idea that guided our work is that the cost should contain a term of energetic consumption. During a movement, such a term is related to the work of muscular forces. However, work is a signed physical quantity that may cancel itself out, even though both active and resistive

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forces consume energy in muscles. Therefore, work has to be always counted positive in order to express the energy expenditure of a movement: this is the *absolute work* of forces. The point is that this function is nondifferentiable and actually presents the typical (Lipschitz) nonsmoothness, the one of the absolute value function. In other words, the simultaneous inactivation of muscles that we observed provides evidence for an absolute-work-like cost. Note that a similar nonsmooth cost function has been proposed by other authors [24].

Reciprocally, we establish what we call the "inactivation principle": minimizing a cost similar to the absolute work implies the presence of simultaneous inactivation of both agonist and antagonistic muscles acting on a joint during fast movements.

To summarize, our analysis shows that inactivation is a kind of *necessary and sufficient condition for the minimization of an absolute-work-like cost*. As far as we know, this is the first time that such a condition has been proved.

The above described methodology is presented in Section 2 in the large framework of fully actuated mechanical systems. The proof of the technical transversality results is postponed to Section 5.

The purpose of Section 3 is to apply the above theory to the problem under consideration, the arm movements. We first report experimental data obtained in a previous work [5]; then an optimality criterion is proposed that allows one to reproduce temporal and spatial features of biological arm movements.

We finally present in Section 4 important extensions of the inactivation principle to more specific systems.

2. METHODOLOGY

Although human vertical arm movements are studied here, our methodology may apply to locomotion, whole-body reaching, and more generally to a very general class of mechanical systems. We present the theory in this larger framework.

2.1. Modelling. *Fully actuated mechanical systems.* We consider a mechanical system with generalized coordinates $x \in \mathbb{R}^n$ (or in an n-dimensional manifold) and Lagrangian:

$$L(x, \dot{x}) = \frac{1}{2}\dot{x}^{\mathrm{T}}M(x)\dot{x} - V(x),$$

where $M(x)$ is the inertia matrix (which we assume to be symmetric and invertible) and $V(x)$ is the potential energy (here due to gravity).

We divide the external generalized forces acting on the system into two components: the first one, denoted by $\tau = S(x)u$, resulting from the input u, and the second one, denoted by $N(x, \dot{x})$, representing any other forces acting on the system, mainly friction forces. Each input u_i represents the action of an actuator (a pair of muscles in the context of this paper) and is generally supposed to be bounded.

The main assumption is that the control acts on every degree of freedom (so the denomination "fully actuated"), that is, $u \in \mathbb{R}^n$ and $S(x)$ is invertible for every x. We will sometimes assume moreover (in the *exactly fully actuated* case) that we directly control each degree of freedom, that is, $S(x) = \mathrm{Id}$. Note that this last assumption is always satisfied up to some feedback of the type $\tau = S(x)u$.

The equations of motion are given by substituting the value of L into Lagrange's equation,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = S(x)u + N(x, \dot{x}) = \tau + N(x, \dot{x}).$$

They are of the form $\ddot{x} = \phi(x, \dot{x}, u)$, with

$$\phi(x, \dot{x}, u) = M(x)^{-1}\big(N(x, \dot{x}) - \nabla V(x) - C(x, \dot{x})\dot{x} + \tau\big), \tag{1}$$

and the Coriolis matrix $C(x, \dot{x}) \in M_n(\mathbb{R})$ is defined as

$$C_{ij}(x, \dot{x}) = \frac{1}{2} \sum_{k=1}^{n} \left(\frac{\partial M_{ij}}{\partial x_k} + \frac{\partial M_{ik}}{\partial x_j} - \frac{\partial M_{kj}}{\partial x_i} \right) \dot{x}_k.$$

To summarize, a *fully actuated mechanical system* is described by an equation of the general form (Σ)

$$\ddot{x} = \phi(x, \dot{x}, u), \tag{2}$$

where we assume that

- x belongs to $\mathbb{R}^n$ (or to an n-dimensional differentiable manifold);
- u belongs to a subset U of $\mathbb{R}^n$ that is a product of intervals of the type

$$U = [u_1^-, u_1^+] \times \ldots \times [u_n^-, u_n^+];$$

  moreover, the origin lies in the interior $\operatorname{int} U$ of U (i.e., $u_i^- < 0 < u_i^+$);
- the mapping ϕ is smooth, i.e., $\phi \in C^\infty(\mathbb{R}^{3n}, \mathbb{R}^n)$, and its Jacobian matrix $\frac{\partial \phi}{\partial u}(x, \dot{x}, u)$ is always invertible.

As a control system, (Σ) writes as

$$\dot{X} = \Phi(X, u), \qquad X \in \mathbb{R}^{2n}, \qquad u \in U \subset \mathbb{R}^n, \tag{3}$$

with $X = (x, y) = (x, \dot{x})$.

Remark 1. The assumption on ϕ implies that system (Σ) is feedback linearizable.

Optimal control problem. Movements between equilibrium points are defined by their duration T and by a pair of initial and final conditions (x_s, x_t) in the configuration space. The system moves from x_s to x_t, starting and ending with zero velocity.

Movements are assumed to be optimal with respect to a certain integral cost of the form

$$J(u) = \int_0^T f(X, u) \, dt. \tag{4}$$

In the paper f is referred to as the *cost function*. The term J is called the *integral cost* or the *optimality criterion*.

We now define our optimal control problem. Consider the controlled system (Σ) in the form of equation (3). Fix a source point $X_s = (x_s, 0) \in \mathbb{R}^{2n}$, a target point $X_t = (x_t, 0) \in \mathbb{R}^{2n}$, and a time $T > 0$.

Then, the optimal control problem is

$(\mathcal{P})$ minimize the integral cost $J(u)$ among all trajectories of (Σ) connecting X_s to X_t in time T.

Remark 2. A simplifying assumption in our work is that the duration T of the motion is fixed. This is not essential, since:

(i) Pontryagin's maximum principle [28] also allows one to deal with free movement durations: the time T is then determined by a supplementary condition of optimality (see [28]);

(ii) as in [15], one could search for the time T that leads to a given amount of the integral cost.

Here, the latter approach is better suited because the optimal cost will be a strictly decreasing function of T [6, Theorem 1].

The following theorem proves that this problem is well-posed.

Theorem 1 (existence of optimal trajectories). *Assume that f is strictly convex with respect to u. Then the minimum of $(\mathcal{P})$ is reached by some optimal trajectory.*

This is shown in [6] in a particular case (the 1-dof arm) and is a consequence of boundedness of the controls and convexity with respect to u of both the cost function and system (Σ). The idea is that a minimizing sequence of trajectories converges for some compactness reason of Ascoli type, and the limit is a trajectory of the system by convexity. General results of this type may be found in [21].

Our aim is to solve the inverse optimal control problem: given system (Σ) and all (or at least many) optimal trajectories of $(\mathcal{P})$, retrieve the cost function f. Of course, the qualitative properties of these optimal trajectories are crucial in the study. We will show that a very important concept for this matter is that of *inactivation*.

Definition 1. A *partial inactivation* (or simply *inactivation*) is an occurrence during a certain strictly positive time interval of an optimal trajectory corresponding to $u_i = 0$ for some i, i.e., the ith control is zero during this time interval. A *total inactivation* is a simultaneous inactivation of all controls.

2.2. Necessity of nonsmoothness. The purpose of this subsection is to show that, for the occurrence of inactivation in optimal trajectories, it is necessary that the minimized integral cost contain a term with some nonsmoothness at $u = 0$ (recall that $u_i = 0$ corresponds to inactivation at the level of the ith degree of freedom).

To emphasize the dependence on the cost, we denote by $(\mathcal{P}_f)$ the optimal control problem $(\mathcal{P})$ associated with a given cost function f.

Theorem 2. *There exists an open and dense subset O of $C^\infty(\mathbb{R}^{3n}, \mathbb{R})$ (endowed with the C^∞ Whitney topology) such that if $f \in O$, then $(\mathcal{P}_f)$ does not admit minimizing controls containing total inactivations, except maybe if the associated trajectory is an equilibrium point of the vector field $\Phi(X, 0)$. In addition, for every integer m, the set O can be chosen so that its complement has codimension larger than m.*

Remark 3. 1. In the previous theorem, we use the Whitney topology on the set of cost functions f to be minimized. It is the usual topology in this setting. If we restrict ourselves to a fixed compact set, it is equivalent to consider the usual topology of C^∞ convergence on this compact set.

2. The fact that the bad set (the set of exceptional cost functions for which inactivation may occur) has codimension infinity (i.e., codimension larger than m, for all m) means that the good set is extremely large.

The proof of this theorem is postponed to Section 5.

The gist of the proof is the following: we assume that the cost function is smooth, and we show that (up to exceptional and unstable cases for the cost) the only optimal trajectories that are constant can be either equilibria trajectories or bang trajectories (i.e., trajectories lying in the boundary of the control set). This is done by using transversality arguments.

Roughly speaking, for inactivation to be optimal in a stable way (i.e., remain optimal while not overly perturbing the cost to be minimized), it is necessary that the cost function f be nonsmooth at $u = 0$.

A similar theorem also holds for partial inactivation. But in that case, for technical reasons, we have to restrict ourselves to the set of all C^∞ functions f that are moreover strictly convex with

respect to u, in the strong sense that the Hessian of f with respect to u is everywhere positive definite. This clearly defines an open subset $\mathcal{SC} \subset C^\infty(\mathbb{R}^{3n}, \mathbb{R})$ for the Whitney topology.

Theorem 3. *There exists an open and dense subset O' of $\mathcal{SC}$ such that if $f \in O'$, then $(\mathcal{P}_f)$ does not admit minimizing controls that contain partial inactivation, except maybe if the associated trajectory on the subinterval is an equilibrium of the system. In addition, for every integer N, the set O' can be chosen so that its complement has codimension greater than N.*

The proof of this more difficult result is given in Section 5.

2.3. Inactivation principle. We will restrict ourselves in this subsection to *exactly fully actuated mechanical systems*. Recall (see Subsection 2.1) that the latter are systems (Σ) such that

$$M(x)\ddot{x} = M(x)\phi(x, \dot{x}, u) = \psi(x, \dot{x}) + u,$$

and that, up to a feedback, every fully actuated mechanical system can be written in this form.

Absolute work. For actuated mechanical systems, the physical quantity that measures energy is the work of forces. However, the work of a force pulling in the direction arbitrarily defined as positive may cancel with the work of the force pulling in the opposite direction. Thus, the *absolute work* measures the energy expenditure of a movement, and more specifically, the absolute work of controlled forces counts the mechanical energy actually spent to control the system.

For an exactly fully actuated mechanical system (Σ), the controlled forces are just u and their work is

$$w = \int u\, dx = \int \sum_{i=1}^{n} u_i\, dx_i = \int \sum_{i=1}^{n} u_i \dot{x}_i\, dt.$$

The *absolute work* Aw of these forces is then defined as

$$Aw = \int \sum_{i=1}^{n} |u_i \dot{x}_i|\, dt. \tag{5}$$

In the coordinates $X = (x, y)$, $\dot{A}w$ is the function

$$\dot{A}w(X, u) = \sum_{i=1}^{n} |u_i y_i|.$$

In view of Subsection 2.2, it is very important to notice that $\dot{A}w$ is Lipschitz continuous, but *nondifferentiable* with respect to u. Indeed, its Clarke's generalized gradient with respect to u_i is

$$\partial_{u_i} \dot{A}w(X, u) = \begin{cases} -|y_i| & \text{if } u_i < 0, \\ [-|y_i|, |y_i|] & \text{if } u_i = 0, \\ |y_i| & \text{if } u_i > 0 \end{cases} \tag{6}$$

and thus is a nonempty interval at a pair (X, u) with $u_i = 0$, $y_i \neq 0$.

We will consider integral costs depending explicitly on the variation $\dot{A}w$ of the absolute work, that is, of the form

$$J(u) = \int_0^T f(X, u)\, dt, \qquad \text{with} \quad f(X, u) = \varphi(\dot{A}w, X, u), \quad \frac{\partial \varphi}{\partial \dot{A}w} \neq 0, \tag{7}$$

where φ is smooth. Due to the absolute work term, the proposed cost function is nonsmooth (nondifferentiable) but Lipschitz continuous with respect to u. This slight difference is crucial in our

study. For technical reasons we will also assume that the cost function $f(X, u) = \varphi(\dot{A}w(X, u), X, u)$ is strictly convex with respect to u (assumption A).

Remark 4. 1. The assumption of strict convexity, although technical, is natural: it implies that the function φ has a unique minimum with respect to u. The weakest possible hypothesis to obtain the inactivation principle (see Theorem 4) is precisely that φ has a unique minimum with respect to u. In that case, the existence of a minimizing trajectory will not be guaranteed (it has to be assumed). Assuming strict convexity is a way to assume both a unique minimum with respect to u and the existence of a minimizing trajectory (see Theorem 1).

2. In fact, the typical nonsmoothness (Lipschitz) is that of the absolute value function. But it can be easily taken into account that "negative work" may cost less than "positive work" (this is the case for the mechanical systems arising from physiology): in place of the function $|u|$, one has to consider the Lipschitz function $\lambda|u|$ for $u > 0$ and $\mu|u|$ for $u \leq 0$. We decided here to limit ourselves to the "nonweighted" absolute work, for the sake of simplicity in exposition.

Pontryagin's maximum principle. Let us first write for our optimal control problem ($\mathcal{P}$) the necessary conditions of optimality given by the maximum principle.

Denote by

$$h(\lambda, X, P, u) = \lambda f(X, u) + P.\Phi(X, u)$$

the Hamiltonian of the problem, where $P \in (\mathbb{R}^n)^*$ (the dual space of $\mathbb{R}^n$) and $\lambda \leq 0$.

If $(X(t), u^*(t))$ is an optimal trajectory of the problem, then there exists an *adjoint vector* $P(t) \in (\mathbb{R}^n)^*$, $P(t)$ being absolutely continuous, $(\lambda, P(t))$ never vanishing, such that

(i) $(X(t), P(t))$ meet the Hamiltonian equations:

$$\dot{X}_i = \frac{\partial h}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial h}{\partial X_i}; \tag{8}$$

(ii) the Hamiltonian $h(\lambda, X(t), P(t), u^*(t))$ reaches its maximum with respect to u at almost every time $t \in [0, T]$.

Since we assume the cost function $f(X, u)$ to be strictly convex with respect to u, condition (ii) can be replaced by (if the maximum is not attained on the constraints)

$$0 \in \partial_u h. \tag{9}$$

Let us recall some classical terminology.

- An *extremal* is a trajectory of the system meeting the necessary conditions provided by the maximum principle.
- A *singular* extremal is an extremal corresponding to $\lambda = 0$ (or equivalently, to the minimum-time problem). Extremals corresponding to $\lambda < 0$ are called *regular*.
- A *bang* extremal is an extremal such that for almost all $t \in [0, T]$ one of the control variables u_i can take the two values $u_i = u_i^-$ or $u_i = u_i^+$ only.
- An *abnormal* extremal is a singular extremal which is not bang.

Note that, since our system is feedback linearizable, it admits no such abnormal extremal. To the best of our knowledge, this fact has been noticed for the first time in [8]. Let us briefly recall its proof.

Set $P = (p, q)$. Then our Hamiltonian h, with $\lambda = 0$, can be rewritten as

$$h(0, X, P, u) = py + q\phi(x, y, u) \tag{10}$$

and is smooth with respect to u.

Along an abnormal extremal, the maximum of h is not attained everywhere on the constraints; therefore, $\frac{\partial h}{\partial u} = 0$ on a nonempty interval I. Since $\frac{\partial \phi}{\partial u}$ is invertible, this condition gives $q(t) = 0$ for all $t \in I$. Differentiating, we find that $\dot{q}(t) = 0$ on I and, by the Hamiltonian equations,

$$0 = \dot{q} = -\frac{\partial h}{\partial y} = p.$$

Then, $p(t)$ also has to be zero on I. This is a contradiction with the fact that $(\lambda, p(t), q(t))$ never vanishes.

Thus there are only two kinds of extremals in our optimal control problem: singular bang extremals, associated with discontinuous controls, and regular extremals, whose controls appear to be continuous, as stated in the following result.

Lemma 1. *The optimal controls $u^*(t)$ corresponding to regular trajectories of $(\mathcal{P})$ are continuous with respect to t.*

Proof. Lemma 11 in [13] states the following. Consider a function $F \colon \mathbb{R}^p \times X \to \mathbb{R}^+$, where X is a manifold and F is continuous, with the additional property that for each compact $K \subset X$, the restriction $F_K = F_{|\mathbb{R}^p \times K}$ is proper. Then, $\varphi(x) = \inf_{v \in \mathbb{R}^p} F(v, x)$ is a well-defined mapping, continuous over X. Examination of the proof of this result shows that it also holds for $F \colon \mathbb{R}^p \times X \to \mathbb{R}$. We apply this lemma to our Hamiltonian h. Due to the assumption of strict convexity for the cost function f and to the fact that $\lambda < 0$, $h(t, u) = h(\lambda, X(t), P(t), u)$ is a strictly concave function of u. Moreover, it is continuous since $X(t)$ and $P(t)$ are continuous functions of t. Let $u(t_0)$ be a discontinuity value of the optimal control $u(t)$. This means that we can find a sequence $t_n \to t_0$ such that $u(t_n) \to \widehat{u} \neq u(t_0)$. Applying the above-mentioned lemma to $-h(t, u)$, where u here is the variable v in the lemma, we find that $t \to -h(t, u(t))$ is a continuous map. But the minimum being unique, this contradicts the assumption $u(t_n) \to \widehat{u} \neq u(t_0)$. $\square$

Statement of the inactivation principle. A rough statement of the inactivation principle is as follows: provided that the total duration T of the motion is not too large (compared to the minimum time $T_{\min}$), there is partial inactivation along an optimal trajectory of the problem $(\mathcal{P})$ associated with a cost $J(u)$ of the form given by equation (7). Moreover, total inactivation may appear in a stable way (stable with respect to small smooth perturbations of the cost, or of the system).

To transform this statement into a theorem, we need precise technical assumptions. We introduce two hypotheses for a trajectory (X, u) defined on $[0, T]$:

(H$_1$) X is a regular extremal (therefore, u is continuous).

(H$_2$) Change of sign for u: a component u_i of the control changes sign at some time $t_c \in \,]0, T[$, while $y_i(t)$ keeps constant sign. This means that there are some times t_1 and t_2, $t_1 < t_c < t_2$, such that $u_i(t_1) u_i(t_2) < 0$ and $y_i(t) \neq 0$ for $t_1 \leq t \leq t_2$.

Theorem 4 (inactivation principle). *Along an optimal trajectory of $(\mathcal{P})$ meeting hypotheses (H$_1$) and (H$_2$) there is partial inactivation. Moreover, some of the regular extremals passing through an arbitrary $X \in \mathbb{R}^{2n}$ have total inactivation.*

Proof. Along an optimal trajectory (X, u^*), the Hamiltonian h of the optimal control problem has to be maximal, which means by equation (9) that $0 \in \partial_{u_i} h$ for all $i = 1, \ldots, n$. However,

$$h(\lambda, X(t), P(t), u^*(t)) = \lambda f(X(t), u^*(t)) + P(t).\Phi(X(t), u^*(t)),$$

and $\lambda < 0$ since we consider regular trajectories only. The maximum condition for the Hamiltonian gives $0 \in \partial_{u_i} h(\lambda, X(t), P(t), u^*(t))$, where

$$\partial_{u_i} h = P.\frac{\partial \Phi}{\partial u_i}(X, u^*) + \lambda \frac{\partial \varphi}{\partial u_i}(\dot{A}w, X, u^*) + \lambda \frac{\partial \varphi}{\partial \dot{A}w}(\dot{A}w, X, u^*)\partial_{u_i}\dot{A}w(X, u^*), \tag{11}$$
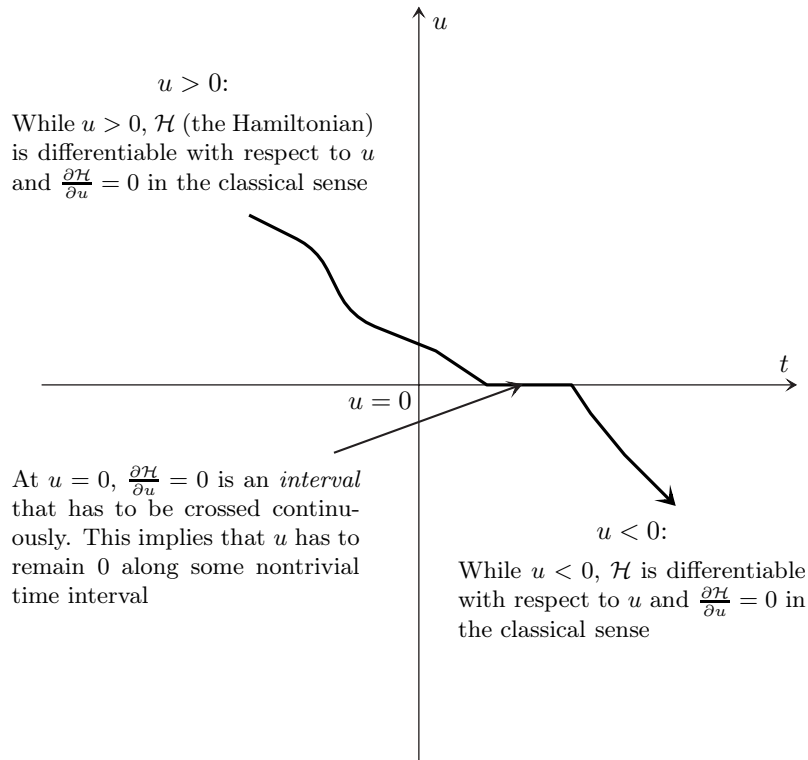
Fig. 1. Intuitive illustration of the proof of the inactivation principle.

the set $\dot{A}w(X, u^*)$ being as in equation (6). By hypothesis (H$_1$) and Lemma 1, the control $u^*(t)$ is continuous. The variables $X(t)$ and $P(t)$ being also continuous, the quantity $\partial_{u_i} h(X(t), P(t), u^*(t))$ is an interval $I(t)$ moving continuously with the time t. This interval degenerates to a point as soon as $u_i^*(t) \neq 0$.

However, since $\frac{\partial \varphi}{\partial Aw}$ and λ are both different from zero, hypothesis (H$_2$) implies that $I(t_c)$ is a nontrivial time interval at the time t_c where $u_i^*(t_c) = 0$. Since $u_i^*(t)$ changes sign at t_c, it takes a certain strictly positive amount of time to cross $I(t_c)$. Then $u_i^*(t)$ remains exactly equal to zero during some nontrivial time interval. This is partial inactivation.

Continuing, we take an arbitrary $X = (x, y)$ with $y_i \neq 0$ for all $i = 1, \ldots, n$ and $\lambda = -1$. Then, for $u = 0$, we compute the set $\mathcal{S} = \partial_u h(X, P, u)$. Setting $P = (p, q)$, we have $\frac{\partial P.\Phi(X,u)}{\partial u_i} = q\frac{\partial \phi(x,y,0)}{\partial u_i}$. Since $\frac{\partial \phi(x,y,0)}{\partial u}$ is an invertible matrix, we can choose q such that 0 is exactly the center of the set $\mathcal{S} \subset \mathbb{R}^n$, which has a nonempty interior. It is clear by construction that the extremals starting from this point $(X, P, 0)$, if continuous, have total inactivation. $\square$

This proof is illustrated in Fig. 1.

Let us examine now the validity of assumptions (H$_1$) and (H$_2$) above.

As for assumption (H$_1$), note that a singular extremal of $(\mathcal{P})$ is an extremal of the minimum-time problem. Thus, when $T > T_{\min}$, assumption (H$_1$) is a consequence of the stronger hypothesis that there is a unique extremal of the minimum-time problem connecting two given points. This last hypothesis is easily checkable and seems to hold true generically. For instance, when (Σ) is linear and $n = 1$, this is proved in [28, Example 1].

Assumption (H$_2$) (the change of sign of the optimal control) is also true in general. This can be proved in the following way.

The input-state mapping

$$P_\Sigma \colon L^\infty_{[0,T],\mathbb{R}^n} \to C^0_{[0,T],\mathbb{R}^{2n}}$$

is continuous for the $*$-weak topology on $L^\infty_{[0,T],\mathbb{R}^n}$ [12]. When $T \to T_{\min}$ from above, we consider the restriction u_T of the optimal control to the interval $[0, T_{\min}]$. This defines a sequence of controls that (by boundedness) we can assume to be $*$-weak convergent to some control $u^*(t)$. By construction, this $u^*(t)$ is a minimum-time control. Since $u_T(t)$ is continuous, if T is close enough to $T_{\min}$, $u_T(t)$ has sign changes close to the sign changes of the minimum-time control $u^*(t)$.

The fact that the minimum-time control $u^*(t)$ has changes of sign can be checked directly, at least when the inertia matrix is constant $M(x) \equiv M$. In this case, the component u_i of the minimum-time control can only commute between the values u_i^- and u_i^+. These values are large enough. Hence, if there is no commutation, u_i is constant and large. Therefore, $\dot{y}_i(t)$ has constant sign and $y_i(t)$ cannot go from zero to zero.

Remark 5. The previous reasoning shows that in general inactivation is located around instants that are close to the instants where the minimum-time control changes sign (commutes). This reasoning also shows that inactivation occurs automatically for a duration T of the motion sufficiently close to the minimum time $T_{\min}$. This is coherent with practical observations showing that for larger T simultaneous inactivation of agonistic and antagonistic muscles may disappear.

3. VALIDATION

3.1. Experimental data. The data were recorded experimentally: they consist of the kinematics and the muscle activity of the arm during one degree of freedom (shoulder rotation) and two degree of freedom (shoulder and elbow rotation) rapid pointing movements in the vertical plane. The description of the experimental apparatus, as well as a complete analysis of the data, is given in [5] (see also [26, 27]). Typical experimental data are given in Fig. 2 for the 1-dof case and in Fig. 3 for the 2-dof one.

First, at a kinematic level, fingertip velocity profiles show significant asymmetries depending on the movement direction and speed, and fingertip paths are slightly curved. It is often considered [14, 26, 29] that relevant indices for these properties are the relative time to peak velocity (TPV), defined as the ratio of the acceleration duration to the total movement duration, and the fingertip path curvature (FPC), defined as the ratio of the maximum path deviation from the line segment connecting the initial and final points of the trajectory to the length of the line segment. In the 1-dof case, since the fingertip path is necessarily a circular arc, the TPV is the only significant measure. Asymmetry means that the TPV is significantly smaller for upward than for downward movements.

Second, at a muscular level, the main feature is the presence of inactivation periods during rapid pointing movements.

For instance, in the 1-dof case, muscle silent phases are noticeable in the EMG signals of Fig. 2. The main flexor and extensor muscles acting on the shoulder joint are simultaneously inactive, so that the net torque resulting from their actions is almost zero during this short period.

For upward movements, simultaneous inactivation of all muscles appeared clearly during a short time interval in the second half of the motion. In some trials, the triceps remained slightly contracted, thus actively maintaining the arm fully extended. For downward movements, inactivation also appeared, although less clearly, during the first half of the movement. This simultaneous inactivation of all muscles lasted on average for approximately 30 ms and was clearly observed in 85% of trials, for upward movements. During this period the arm was almost in free fall, an energetically costless movement. Notably, the activities of all muscles stopped at the same instant. This synchronization suggests that muscle inactivation results from an active optimal motor strategy.

In a typical muscular pattern for the 2-dof case (such as the one depicted in Fig. 3), simultaneous inactivation of pairs of muscles acting on each joint also occurred. Notice the lag between the inactivation at the elbow joint and at the shoulder joint, illustrating that in the 2-dof case the inactivation occurred at each joint separately.
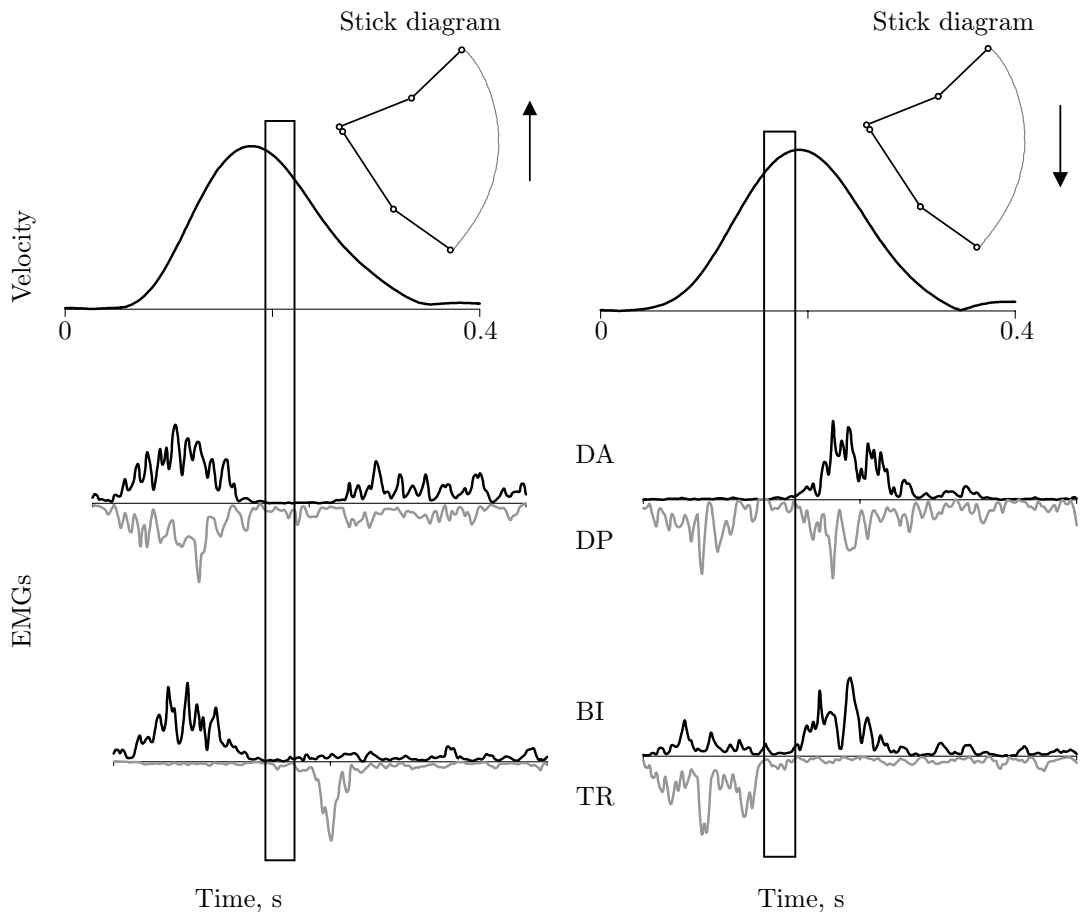
Fig. 2. Typical experimental data of a 1-dof arm motion performed in upward (left) and downward (right) directions. Finger velocity profiles (upper part) and four electromyographic signals (DA, Deltoid (Anterior); DP, Deltoid (Posterior); BI, Biceps; and TR, Triceps) are reported. The periods of muscular inactivation are emphasized by means of rectangular frames. Data are amplitude normalized, and the horizontal axis denotes time (in seconds).

The appearance of simultaneous inactivation was also checked in movements starting from different initial arm configurations.

3.2. The proposed criterion. The mechanical models we use to describe the vertical movements of an arm in the 1-dof and 2-dof cases are the following.

The one degree of freedom arm. The control system is

$$\ddot{x} = u - k\cos x + b\dot{x},$$

where the constant k reflects the action of the gravity field, b is a friction term, and $u \in \mathbb{R}$ is the net torque acting at the joint, u being bounded ($u^- \le u \le u^+$ with $u^- < 0 < u^+$).

The two degree of freedom arm. The mechanical equation is

$$H(\theta)\ddot{\theta} - \widehat{h}(\theta)r(\dot{\theta}) + G(\theta) + B\dot{\theta} = \tau, \tag{12}$$

in which θ belongs to $\mathbb{R}^2$, H is the (symmetric positive definite) matrix of principal inertia moments, $\widehat{h}(\theta)r(\dot{\theta})$ is the Coriolis term, G is the vector of gravitational torques, and B is the matrix of friction terms (a constant here). The term τ is the vector of generalized external torques (the controls in our case), i.e., $\tau = u$.

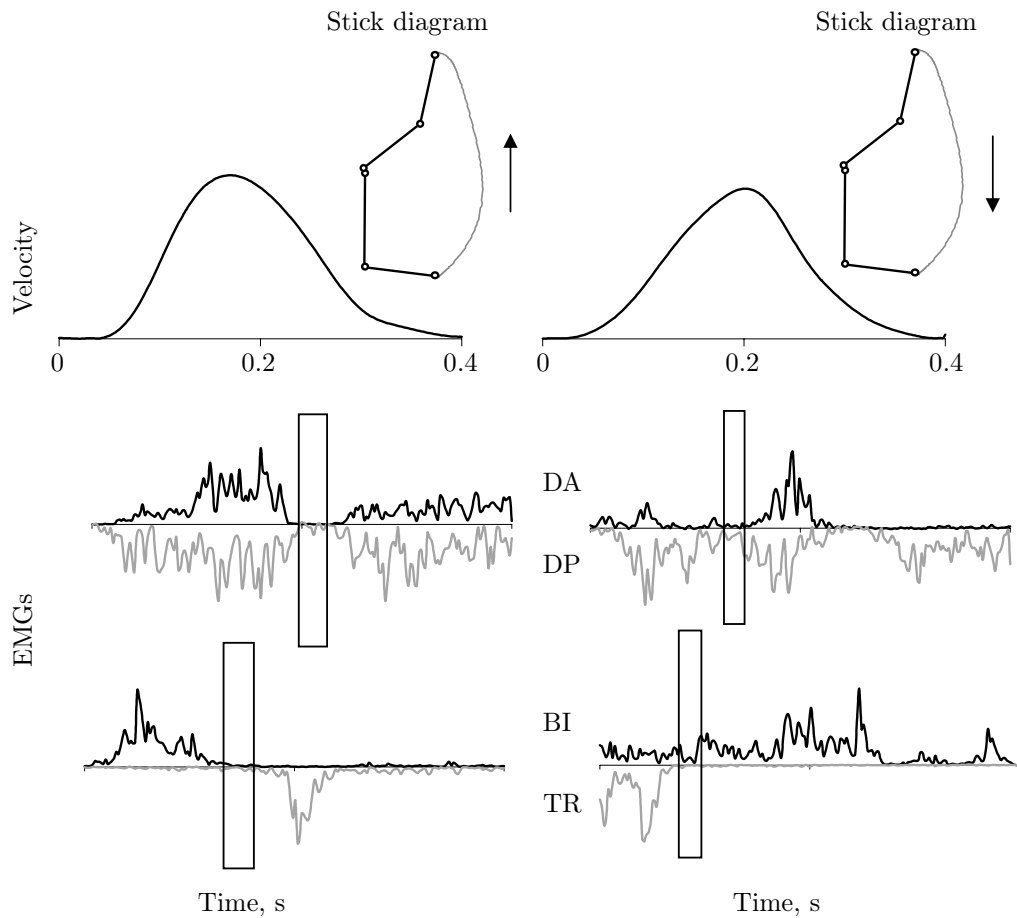Both models are (exactly) fully actuated mechanical systems.

Fig. 3. Typical experimental data of a 2-dof arm movement. Same notations as in Fig. 2.

The main purpose here is to infer a criterion to be minimized. Following Section 2, from the presence of inactivations in the experimental data one concludes that the cost contains a term similar to the absolute work. Reciprocally, such a term will ensure the appearance of inactivations. We then propose an integral cost in the general form:

$$J(u) = \int_0^T \widetilde{f}(x, y, u) \, dt + Aw. \tag{13}$$

This expression represents a compromise between the absolute work Aw and something like a "comfort term" defined by the function $\widetilde{f}$. For instance (nonexhaustive list), one may choose the acceleration squared (as in [3]) or the torque squared (as in [24]) for the function $\widetilde{f}$.

This additional term is not crucial. One could assume that the CNS only minimizes the absolute work, but it seems to also minimize some integral costs accounting for the smoothness or precision of the movements [3, 11, 17, 32]. While the definition of the mechanical energy spent is well established, what should be the comfort term is more subjective. It may suggest that the motor system would avoid large accelerations, so as not to expose tendons and articulations to large jerks.

Here, in all examples and simulations, we will assume that $\widetilde{f}$ is proportional to the energy of acceleration (in the sense of signal processing), that is,

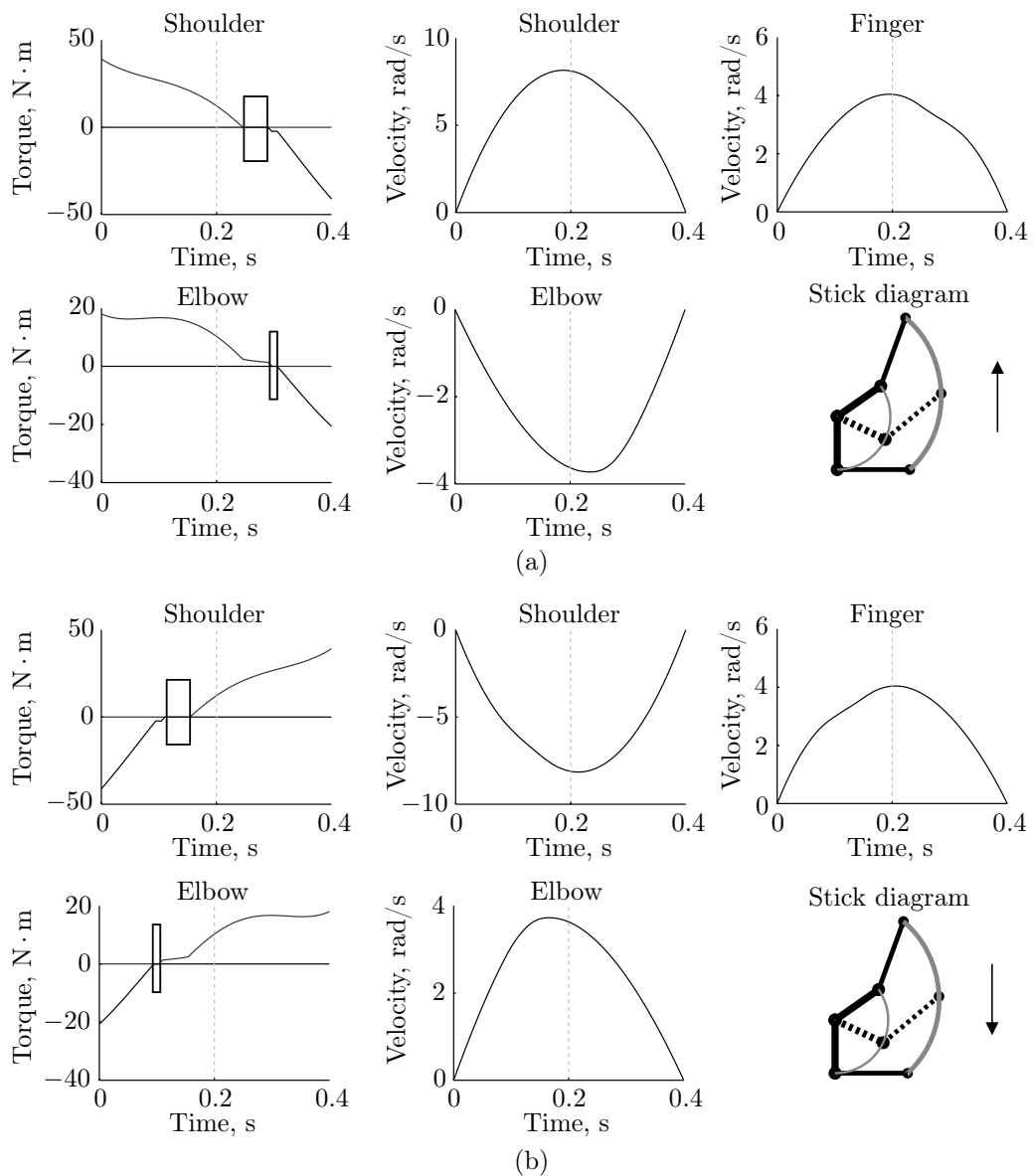$$\widetilde{f}(X, u) = \sum_{i=1}^n \alpha_i (\ddot{x}_i)^2. \tag{14}$$

Fig. 4. Results for a simulated 2-dof arm movement: (a) upward direction; (b) downward direction. Torques and angular velocities, respectively denoted u (N·m) and y (rad/s), are plotted with respect to time (seconds), along with the finger velocity (m/s).

The weighting parameters $(\alpha_i)_{i=1,\dots,n}$ are strictly positive constants. We take the values $\alpha_1 = 0.25$ and $\alpha_1 = 0.25$, $\alpha_2 = 0.25$ in the 1-dof and 2-dof cases, respectively.

Nevertheless, we also simulated movements with weighting parameters ranged between 0.05 and 1, and all these instances of the model lead to plausible movements. Therefore, these parameters may be considered as tuning parameters to improve the quantitative fitting of the model to each participant.

Note that this term $\widetilde{f}(X, u)$ is strictly convex with respect to u (in accordance with assumption A, see Subsection 2.3).

3.3. Validation of the model. The optimal problem $(\mathcal{P})$ corresponding to minimizing the compromise of equation (13) in both 1-dof and 2-dof cases is studied in detail in [6] and [5]. We show in Fig. 4 a simulated 2-dof movement, which illustrates the theoretical results (see Subsection 4.1 for simulated 1-dof movements).

It appears that, for the set of movements and conditions tested, the proposed criterion seems to be well suited. Notably, we retrieve in all instances of the model the two main features of the experimental data.

Firstly, in accordance with the inactivation principle, a *partial inactivation* period (emphasized by a rectangular frame in Fig. 4) is observed at each joint slightly after the time of peak velocity during an upward movement, and slightly before the time of peak velocity during a downward movement. This is in agreement with corresponding experimental results (see Fig. 3).

It is worth noticing moreover that simultaneous inactivation disappears if movements are too slow or too small. Again, this is consistent with experiments.

Secondly, fingertip velocity profiles are *asymmetric* (the acceleration duration is shorter than the deceleration duration) and fingertip paths are curved. For instance, under the conditions of Fig. 4, the relative time to peak velocity (TPV) is equal to 0.46 and 0.54 for upward (U) and downward (D) directions, respectively, consistent with the experimental values 0.42 ± 0.02 and 0.53 ± 0.04. The fingertip path curvature (FPC) is equal to 0.20, which is close to the value 0.14 ± 0.04 experimentally recorded.

Note also that since the response time of muscles was not modelled in this case, jumps on the joint torques occur at the initial and final times, leading to nonzero accelerations on the corresponding velocity profiles. We will see how to handle this problem in the next section.

To be complete, let us mention that experiments and simulations have been realized very recently in the 3-dof case that confirm the relevancy of the theoretical model [5].

4. EXTENSIONS OF THE INACTIVATION PRINCIPLE

The inactivation principle is valid for much more detailed models than the one presented above. We will show in this section that it applies, in particular, in two important cases. Firstly, it holds when considering that the net torque actually comes from agonistic and antagonistic torques. Secondly, this principle also holds when assuming that the torques are produced by muscles with nonzero response times, i.e., when the torques cannot immediately reach their maximum value. For instance, when the control is the derivative of torques (called a gradient constraint case) or when the dynamics of muscles is modelled, the inactivation period is still present. These results are crucial for interpreting the inactivation on net torques as simultaneous inactivation of both agonistic and antagonistic muscles in practice.

4.1. The case of gradient constraints on the torques. This is a first extension of the theory. We explain what happens in the 1-dof case only, but the case of two degrees of freedom is similar.

The control system is the one of the 1-dof case, but we require moreover that the derivative of the torque u be bounded.

We introduce a new control $v = \dot{u}$, and the problem may be rewritten as (we neglect the friction term)

$$\dot{x} = y, \qquad \dot{y} = u - k\cos x, \qquad \dot{u} = v,$$

$$v^- \le v \le v^+, \qquad v^- < 0, \qquad v^+ > 0,$$

$$J(v) = Aw + \int_0^T \widetilde{f}(X, u)\, dt \to \min_v.$$

Now the cost function is not differentiable anymore with respect to the state (in place of the control in previous sections). Therefore, Clarke's nonsmooth version of the maximum principle is needed [10].
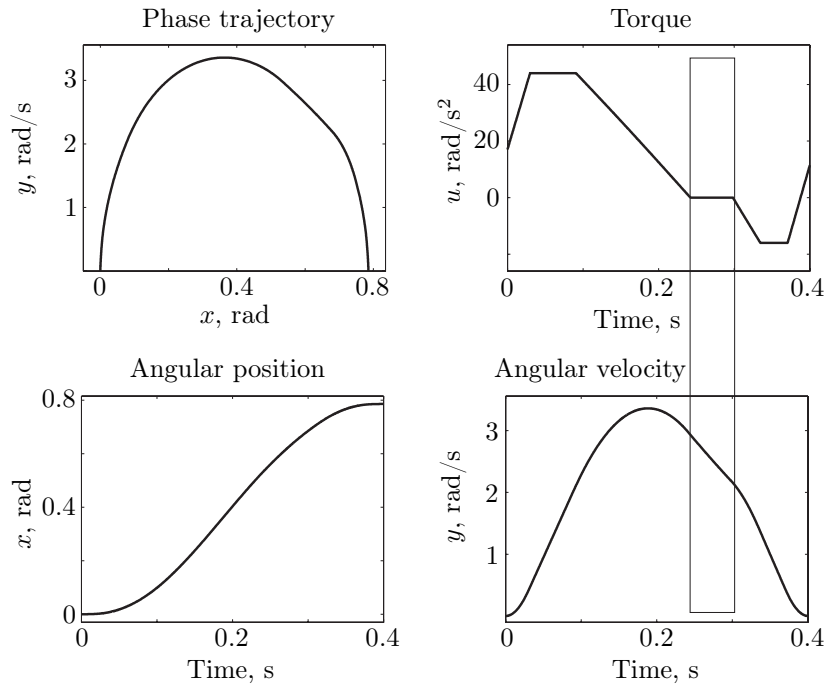
Fig. 5. Results for a simulated 1-dof upward movement, with gradient constraints on the torque. The theoretical phase of inactivation of the muscles is shown (rectangular frame). Note that the time to peak velocity (TPV) is 0.47 in this case. It would be equal to 0.53 for the corresponding downward movement, according to experimental findings showing the same directional asymmetries.

If (p, q, r) denotes now the adjoint variables, we get

$$\widetilde{H} = \lambda\big(y|u| + \widetilde{f}(x, y, u)\big) + py + q(u - k\cos x) + rv.$$

Once again, x, y, p, q, r, u are continuous (by nature now, just as classical solutions of differential equations). The *a priori* fact that y remains positive is just checked numerically. However, it is expectable from the results obtained without gradient constraints on the torques.

Also, for reasons similar to those of Subsection 2.3, the abnormal extremals may be excluded: the maximality of the Hamiltonian for nonbang trajectories implies that r is identically zero, which implies, with two successive differentiations, that q and p, respectively, are also identically zero. The total adjoint vector is zero, which contradicts the maximum principle. Hence we may assume $\lambda = -1$.

We assume that the gradient constraints v^- and v^+ are large enough for the optimal control to be of the following type: the gradient constraints are active only at the beginning and at the end of the motion. If we refer to the scenario occurring in the 1-dof case, this should be what happens: without the gradient constraints, the gradient is never large. Then, there will be saturation of the gradient constraints only because of the jumps at the beginning and at the end of the motion. Numerical computations confirm this scenario, as illustrated in Fig. 5.

For instance, consider that $x_t > x_s$, i.e., an upward movement. Then, to connect (in an optimal way) the source $(x_s, 0, u_s)$ to the target $(x_t, 0, u_t)$, where u_s and u_t are the stationary torques corresponding to the equilibrium positions x_s and x_t, respectively, the strategy must be as follows: $v = v^+$ for $0 \leq t < T_1$; $v^- < v < v^+$ for $T_1 \leq t \leq T_2$; $v = v^-$ for $T_2 < t \leq T$.

Therefore, inside the interval $[T_1, T_2]$, the Hamiltonian is maximal with respect to v and we must have $r(t) = 0$. Therefore, $\frac{dr}{dt} = 0$. But by Clarke's maximum principle this means that $\frac{dr}{dt} \in -\partial_u \widetilde{H} = yI + \frac{\partial \widetilde{f}}{\partial u} - q$, where I is the Clarke gradient of the absolute value function at zero, i.e., $I = [-1, 1]$.

Since $\frac{dr}{dt} = 0$, we conclude:

$$0 \in -\partial_u \widetilde{H} = yI + \frac{\partial \widetilde{f}}{\partial u} - q.$$

This equation was exactly the cause of the presence of inactivation when we proved the inactivation principle (see the proof of Theorem 4).

Therefore, the inactivation phenomenon persists under torque gradient constraints.

Notice that adding gradient constraints also permits getting smoother velocity profiles with zero acceleration at the initial and final times.

4.2. Agonistic–antagonistic actuators. In biomechanical systems, a controlled force generally results from the action of two opposing muscles (one agonistic and one antagonistic). The control system is still, as in Subsection 2.3, a fully actuated mechanical system; however, the applied torque writes now as $u = u_1 - u_2$, where u_1 and u_2 are the agonistic and antagonistic torque, respectively, and are subject to the constraint $0 \le u_{1i} \le u_i^+$ and $0 \le u_{2i} \le -u_i^-$ for $i = 1, \ldots, n$.

The purpose here is to show that the inactivation principle persists in this situation.

For the net torque u, the cost is the compromise between the absolute work and a smooth term of equation (13),

$$J(u) = \int_0^T \widetilde{f}(x, y, u)\, dt + Aw, \qquad \text{with} \quad Aw = \int_0^T \sum_{i=1}^n |u_i y_i|\, dt.$$

This means that, for agonistic–antagonistic torques, we shall minimize

$$J'(u_1, u_2) = \int_0^T \widetilde{f}(x, y, u_1 - u_2)\, dt + Aw',$$

where Aw' is the total absolute work of external torques:

$$Aw' = \int_0^T \left(\sum_{i=1}^n |u_{1i} y_i| + \sum_{i=1}^n |u_{2i} y_i| \right) dt.$$

Consider first a pair (u_1, u_2) minimizing J' and denote by $J'^* = J'(u_1, u_2)$ the optimal value of the cost. We set $u = u_1 - u_2$. Clearly, u applied to the system

$$\ddot{x} = \phi(x, \dot{x}, u) \tag{15}$$

and u_1, u_2 applied to the system

$$\ddot{x} = \phi(x, \dot{x}, u_1 - u_2) \tag{16}$$

produce identical x-trajectories.

Therefore,

$$J(u) = \int_0^T \left(\widetilde{f}(x, y, u_1 - u_2) + \sum_{i=1}^n |(u_{1i} - u_{2i}) y_i| \right) dt$$

$$\le \int_0^T \left(\widetilde{f}(x, y, u_1 - u_2) + \sum_{i=1}^n |u_{1i} y_i| + \sum_{i=1}^n |u_{2i} y_i| \right) dt = J'(u_1, u_2) = J'^*.$$

This shows that the minimum $J^* = \min_u J(u) \le J'^*$.

Conversely, assume that u provides the minimum J^* of $J(u)$. We define u_1 and u_2 from u as follows (for $i = 1, \ldots, n$):

$$u_{1i}(t) = \begin{cases} u_i(t) & \text{if } u_i(t) > 0, \\ 0 & \text{elsewhere} \end{cases} \quad \text{and} \quad u_{2i}(t) = \begin{cases} -u_i(t) & \text{if } u_i(t) < 0, \\ 0 & \text{elsewhere.} \end{cases} \tag{17}$$

Again $u_1 - u_2 = u$. Hence applying u to equation (15) produces the same x-trajectory as applying $u_1 - u_2$ to equation (16). Therefore, by the definition of u_1 and u_2, we have

$$J'(u_1, u_2) = \int_0^T \left(\widetilde{f}(x, y, u_1 - u_2) + \sum_{i=1}^n (|u_{1i}y_i| + |u_{2i}y_i|) \right) dt = \int_0^T \left(\widetilde{f}(x, y, u_1 - u_2) + \sum_{i=1}^n |(u_{1i} - u_{2i})y_i| \right) dt.$$

This means that

$$J'(u_1, u_2) = J^*, \tag{18}$$

which implies that $J'^* \leq J^*$. It is now clear that $J'^* = J^*$, and also by equation (18) the minimum is reached by u_1, u_2 defined in (17).

Notably, by construction, the torques u_{1i} and u_{2i} have simultaneous inactivation only when $u_i = 0$, for $i = 1, \ldots, n$.

We have proved the following theorem.

Theorem 5 (simultaneous inactivation for agonistic–antagonistic torques)**.** *In the case of agonistic–antagonistic torques, minimizing a cost containing the absolute work leads to simultaneous inactivation of both torques, exactly at the same times when the optimal net torque is inactive.*

4.3. Dynamics of the muscles and the triphasic pattern. In this subsection, we consider, as in Subsection 4.2, agonistic–antagonistic torques, but we assume some dynamics on each muscle. For the sake of simplicity, we assume first order dynamics on the muscles, but this restriction is not crucial. Also, we present the results in the 1-dof case and, in order to make the computations more tractable, we make the small angles assumption: the system is then replaced by its linearization.

Then, adding the first order time constants σ_1 and σ_2 on both muscles, we get the following control system:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = u_1 - u_2 - k, \\ \dot{u}_1 = -\dfrac{u_1}{\sigma_1} + v_1, \\ \dot{u}_2 = -\dfrac{u_2}{\sigma_2} + v_2 \end{cases} \tag{19}$$

with $v_1, v_2 \geq 0$.

The minimization problem is

$$\min_{v_1, v_2} \int_0^T \left(yu_1 + yu_2 + \alpha \dot{y}^2 \right) dt.$$

We will use the a priori fact (which is checked numerically) that, as in the case of the torque control, y remains positive during the upward motion [6]. The Hamiltonian may be written as

$$\mathcal{H} = -y(u_1 + u_2) - \alpha(u_1 - u_2 - k)^2 + py + q(u_1 - u_2 - k) + r_1\left(-\frac{u_1}{\sigma_1} + v_1\right) + r_2\left(-\frac{u_2}{\sigma_2} + v_2\right).$$

At this point, there is an important technical detail that physiologically makes sense. It can be understood as muscular co-activation at the end of the motion, a well-known phenomenon in physiology.
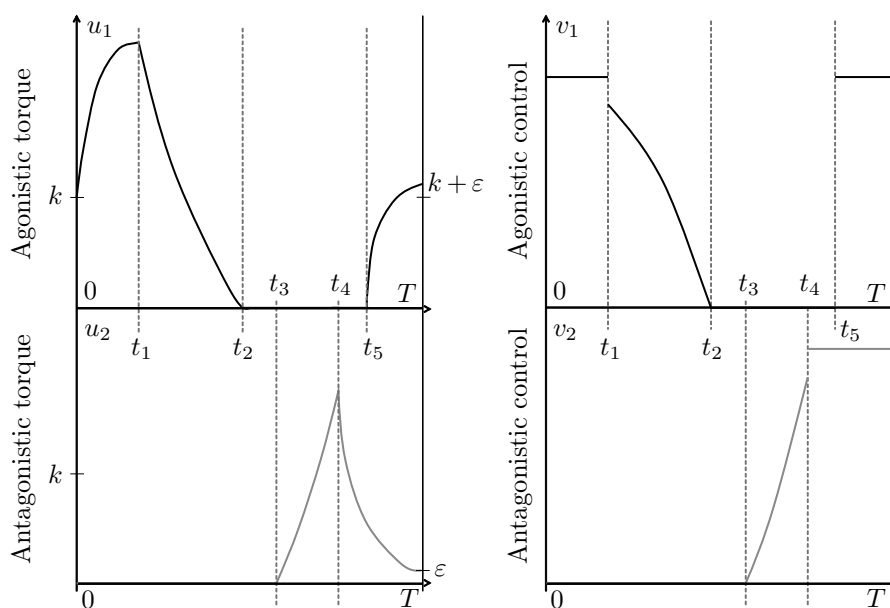
Fig. 6. Optimal triphasic pattern: Illustration of the optimal behaviour of a 1-dof arm, under the small angles assumption and with a pair of agonistic and antagonistic muscles modelled by first-order dynamics. The subscripts 1 and 2 denote the flexor and extensor muscles, respectively. The triphasic pattern is an agonistic burst, followed by an antagonistic burst, and again an agonistic burst. The inactivation occurs between the first agonistic and antagonistic bursts. The times t_i denote the commutation times. The left graphs illustrate the behaviour of the angular torques (u). The right graphs illustrate the behaviour of the control signals (v), which are the input signals for muscles contractions (i.e., the signals driven by motoneurons). All signals are plotted with respect to time t varying between 0 and T.

Due to the first order linear dynamics on the muscles and the constraints $u_i \geq 0$, we can only go back to zero asymptotically. Therefore, the terminal condition $u_2^{\mathrm{t}} = 0$ is impossible, i.e., the antagonistic torque cannot go back to exactly zero at the end of the movement.

Hence we require, with $\varepsilon > 0$,

$$\text{(I)} \quad u_1^{\mathrm{s}} = k \quad \text{and} \quad u_2^{\mathrm{s}} = 0, \qquad \text{(II)} \quad u_1^{\mathrm{t}} = k + \varepsilon \quad \text{and} \quad u_2^{\mathrm{t}} = \varepsilon. \tag{20}$$

Notice that when modelling muscles dynamics, the initial and final values of both agonistic and antagonistic torques must be specified in order to maintain the arm at equilibrium.

Requirement (II) is the co-activation at the terminal time T. Then, explicit computations with the maximum principle, together with a numerical research of the commutation times, show that the optimal scenario is as shown in Fig. 6.

One can recognize the classical scenario called "triphasic pattern" [16], namely, an agonistic burst followed by an antagonistic burst followed again by an agonistic burst (the scenario ends with the above-mentioned co-contraction of the muscles).

In fact, our theory shows that it should be called "quadriphasic pattern" since there is an inactivation interval between the first agonistic pulse and the antagonistic one.

5. PROOF OF THEOREMS 2 AND 3

Proof of Theorem 2. The proof is based on Thom's transversality theorem. We will then make the computations in the spaces of jets. For a positive integer m and a pair $(X, u) \in \mathbb{R}^{2n} \times \mathbb{R}^n$, we denote by $\mathcal{J}_{(X,u)}^m$ the space of m-jets at (X, u) of functions in $C^\infty(\mathbb{R}^{3n}, \mathbb{R})$.

Set $F(X) = \Phi(X, 0)$ and fix a point $X^0 \in \mathbb{R}^{2n}$ which is not an equilibrium of the vector field F. We define $\mathcal{A}^m(X^0) \subset \mathcal{J}^m_{(X^0,0)}$ as the set of m-jets of functions $f \in C^\infty(\mathbb{R}^{3n}, \mathbb{R})$ such that the trajectory of equation (3) issued from X^0 and associated with the control $u = 0$ is locally minimizing for the optimal control problem $(\mathcal{P}_f)$.

Lemma 2. $\mathcal{A}^m(X^0)$ *is contained in a vector subspace of* $\mathcal{J}^m_{(X^0,0)}$ *of codimension* $n(m - 2)$.

Proof. Without lack of generality we assume $X^0 = 0$. Let $j^m_0 f$ be an m-jet in $\mathcal{A}^m(0)$. By the definition of $\mathcal{A}^m(0)$, the trajectory $X(\cdot)$ of F issued from 0 minimizes the problem $(\mathcal{P}_f)$ on an interval $I = [0, s]$. Thus $X(\cdot)$ satisfies Pontryagin's maximum principle on I: there exists a smooth function $P = (p, q) \colon I \to \mathbb{R}^n \times \mathbb{R}^n$ (the smoothness of P results from that of X) and $\lambda \leq 0$ such that, for all $t \in I$, $(P(t), \lambda) \neq 0$ and

(P1) $\dot{P}(t)^{\mathrm{T}} = -\frac{\partial H}{\partial X}(X(t), P(t), \lambda, 0)$,

(P2) $H(X(t), P(t), \lambda, 0) = \max_{v \in U} H(X(t), P(t), \lambda, v)$,

where the Hamiltonian of the problem is

$$H(X, P, \lambda, u) = p^{\mathrm{T}}y + q^{\mathrm{T}}\phi(X, u) + \lambda f(X, u).$$

Note that since $0 \in \operatorname{int} U$, property (P2) implies $\frac{\partial H}{\partial u}(X(t), P(t), \lambda, 0) = 0$. It follows that

$$q(t)^{\mathrm{T}} = -\lambda \frac{\partial f}{\partial u}(X(t), 0)\frac{\partial \phi}{\partial u}(X(t), 0)^{-1}.$$

If $\lambda = 0$, then $q \equiv 0$. From $\dot{q} \equiv 0$ and (P1) we deduce $p \equiv 0$ and then $(P, \lambda) \equiv 0$, which is impossible. Thus λ is negative and a standard argument of homogeneity allows normalizing it to $\lambda = -1$. Finally, by virtue of (P1) and (P2), respectively, the following holds on the interval I:

$$\dot{p}^{\mathrm{T}} = -q^{\mathrm{T}}\frac{\partial \phi}{\partial x}(X, 0) + \frac{\partial f}{\partial x}(X, 0),$$

$$\dot{q}^{\mathrm{T}} = -p^{\mathrm{T}} - q^{\mathrm{T}}\frac{\partial \phi}{\partial y}(X, 0) + \frac{\partial f}{\partial y}(X, 0), \tag{21}$$

and

$$q^{\mathrm{T}} = \frac{\partial f}{\partial u}(X, 0)\frac{\partial \phi}{\partial u}(X, 0)^{-1}. \tag{22}$$

Now, recall that on I the dynamics is $\dot{X} = F(X)$. Since $X^0 = 0$ is not an equilibrium point of F, we assume, up to a local change of the coordinates $X = (X_1, \ldots, X_{2n})$ on $\mathbb{R}^{2n}$, that $F = \frac{\partial}{\partial X_1}$. Differentiating equation (21) with respect to time leads to

$$\ddot{q}^{\mathrm{T}} = -\dot{p}^{\mathrm{T}} - \dot{q}^{\mathrm{T}}\frac{\partial \phi}{\partial y} - q^{\mathrm{T}}\frac{\partial}{\partial X_1}\frac{\partial \phi}{\partial y} + \frac{\partial}{\partial X_1}\frac{\partial f}{\partial y} = q^{\mathrm{T}}\frac{\partial \phi}{\partial x} - \frac{\partial f}{\partial x} - \dot{q}^{\mathrm{T}}\frac{\partial \phi}{\partial y} - q^{\mathrm{T}}\frac{\partial}{\partial X_1}\frac{\partial \phi}{\partial y} + \frac{\partial}{\partial X_1}\frac{\partial f}{\partial y}, \tag{23}$$

in which we omit the evaluation at $(X, 0)$.

On the other hand, we can also obtain $\dot{q}^{\mathrm{T}}$ and $\ddot{q}^{\mathrm{T}}$ by differentiating equation (22):

$$\dot{q}^{\mathrm{T}} = \frac{\partial}{\partial X_1}\frac{\partial f}{\partial u} \times \left(\frac{\partial \phi}{\partial u}\right)^{-1} + \frac{\partial f}{\partial u} \times \frac{\partial}{\partial X_1}\left(\frac{\partial \phi}{\partial u}\right)^{-1},$$

$$\ddot{q}^{\mathrm{T}} = \frac{\partial^2}{\partial X_1^2}\frac{\partial f}{\partial u} \times \left(\frac{\partial \phi}{\partial u}\right)^{-1} + 2\frac{\partial}{\partial X_1}\frac{\partial f}{\partial u} \times \frac{\partial}{\partial X_1}\left(\frac{\partial \phi}{\partial u}\right)^{-1} + \frac{\partial f}{\partial u} \times \frac{\partial^2}{\partial X_1^2}\left(\frac{\partial \phi}{\partial u}\right)^{-1}.$$

Substituting these expressions and equation (22) into equation (23), we eliminate q^{T}, $\dot{q}^{\mathrm{T}}$, and $\ddot{q}^{\mathrm{T}}$ and obtain

$$\frac{\partial^2}{\partial X_1^2}\frac{\partial f}{\partial u} + R_X\left(\frac{\partial}{\partial X_1}\frac{\partial f}{\partial u}, \frac{\partial f}{\partial u}, \frac{\partial}{\partial X_1}\frac{\partial f}{\partial X_i}, \frac{\partial f}{\partial X_i}\right) = 0 \qquad \text{on } I,$$

where, for every X, R_X is a linear mapping and $X \mapsto R_X$ is smooth. Successive derivations and evaluation of the derivatives at $t = 0$ (recall that $X(0) = 0$) lead to a system of equations of the form

$$\frac{\partial^k}{\partial X_1^k}\frac{\partial f}{\partial u}(0) + R^k\left(\frac{\partial^j}{\partial X_1^j}\frac{\partial f}{\partial u}(0), \frac{\partial^j}{\partial X_1^j}\frac{\partial f}{\partial X_i}(0); \ j < k, \ 1 \le i \le 2n\right) = 0, \qquad k \ge 2,$$

where each R^k is a linear mapping.

Thus we have proved $\mathcal{A}^m(0) \subset \ker\pi$, where $\pi\colon \mathcal{J}_0^m \to \mathbb{R}^{n(m-2)}$ is the linear mapping which associates

$$\left(\frac{\partial^k}{\partial X_1^k}\frac{\partial f}{\partial u}(0) + R^k\left(\frac{\partial^j}{\partial X_1^j}\frac{\partial f}{\partial u}(0), \frac{\partial^j}{\partial X_1^j}\frac{\partial f}{\partial X_i}(0); \ j < k, \ 1 \le i \le 2n\right)\right)_{2 \le k \le m-1}$$

with an m-jet $j_0^m f$.

This linear mapping being obviously surjective, the conclusion follows. $\square$

Theorem 2 follows from Lemma 2 combined with the classical Thom's transversality theorem. $\square$

Remark 6. In the computations in the jet space, only $f(X,0)$, $\frac{\partial f}{\partial u}(X,0)$, and their derivatives with respect to X appear. Thus the statement of Theorem 2 still holds if we replace $C^\infty(\mathbb{R}^{3n}, \mathbb{R})$ by the set of polynomial functions of u with coefficients in $C^\infty(\mathbb{R}^{2n}, \mathbb{R})$, or, even better, by the space of functions $f(X,u)$ differentiable with respect to u at $u = 0$ (and such that $f(X,0)$ and $\frac{\partial f}{\partial u}(X,0)$ are smooth). On the other hand, since the set O is open, it is also possible to replace $C^\infty(\mathbb{R}^{3n}, \mathbb{R})$ by any of its open subsets, for instance by the set $\mathcal{SC}$.

Proof of Theorem 3. Recall that our control system is of the form

$$\ddot{x} = \phi(x, \dot{x}, u) = \psi(x, \dot{x}) + S(x)u,$$

where the $n \times n$ matrix $S(x)$ is always invertible and $x \mapsto S(x)$ is C^∞.

Setting $X = (x, y)$, we rewrite the system as

$$\dot{X} = F(X) + \sum_{i=1}^n u_i b_i(X), \qquad X \in \mathbb{R}^{2n}, \quad u \in U \subset \mathbb{R}^n, \tag{24}$$

where F and $b_1, \ldots, b_n$ are vector fields on $\mathbb{R}^{2n}$.

An equilibrium of this system is a stationary trajectory $X \equiv X^0$ associated with a control $u \equiv u^0$ such that

$$F(X^0) + \sum_i u_i^0 b_i(X^0) = 0.$$

As the preceding one, the proof is based on Thom's transversality theorem. We will then make the computations in the spaces of jets. For a positive integer N and a pair $(X, u) \in \mathbb{R}^{2n} \times \mathbb{R}^n$, we denote by $\mathcal{J}_{(X,u)}^N$ the space of N-jets at (X, u) of functions in $C^\infty(\mathbb{R}^{3n}, \mathbb{R})$.

Lemma 3. *Let $f \in \mathcal{SC}$. Assume that the trajectory (X, u) minimizing $(\mathcal{P}_f)$ satisfies, on a subinterval I of $[0, T]$,*

- *$u_{i_0} \equiv 0$ for some $i_0 \in \{1, \ldots, n\}$;*
- *$\dot{X} \ne 0$ (i.e., the restriction $X_{|I}$ contains no equilibrium of the system).*

Then there exists $t \in I$ such that the N-jet $j^N_{(X(t),u(t))}f$ belongs to a semi-algebraic subset of $\mathcal{J}^N_{(X(t),u(t))}$ of codimension greater than $N - 2n$.

Proof. Recall that, under the hypothesis of the lemma, there is a trajectory (X, u) minimizing $(\mathcal{P}_f)$. Moreover, this trajectory is not the projection of a singular extremal, and its associated control u is continuous. Thus, by Pontryagin's maximum principle applied on I, there exists a C^1 function $P = (p, q) \colon I \to \mathbb{R}^n \times \mathbb{R}^n$ such that, for all $t \in I$,

(P1) $\dot{P}(t)^{\mathrm{T}} = -\frac{\partial H}{\partial X}(X(t), P(t), u(t))$,

(P2) $H(X(t), P(t), u(t)) = \max_{v \in U} H(X(t), P(t), v)$,

where H is the normal Hamiltonian of the problem,

$$H(X, P, u) = p^{\mathrm{T}}y + q^{\mathrm{T}}(\psi(X) + S(x)u) - f(X, u).$$

By virtue of (P1), the following holds on the interval I:

$$\begin{cases} \dot{p}^{\mathrm{T}} = -q^{\mathrm{T}}\left(\dfrac{\partial \psi}{\partial x}(X) + \dfrac{\partial S}{\partial x}(x)u\right) + \dfrac{\partial f}{\partial x}(X, u), \\[2mm] \dot{q}^{\mathrm{T}} = -p^{\mathrm{T}} - q^{\mathrm{T}}\dfrac{\partial \psi}{\partial y}(X) + \dfrac{\partial f}{\partial y}(X, u). \end{cases} \tag{25}$$

On the other hand, (P2) implies that, for every $t \in I$, $u(t)$ satisfies the Karush–Kuhn–Tucker conditions: there exist Lagrange multipliers $\lambda^+(t)$ and $\lambda^-(t)$ in $\mathbb{R}^n$ such that

$$\begin{cases} S(x(t))^{\mathrm{T}}q(t) - \dfrac{\partial f}{\partial u}(X(t), u(t))^{\mathrm{T}} - \lambda^+(t) - \lambda^-(t) = 0, \\[2mm] \lambda^+_i(t), \lambda^-_i(t) \geq 0, \quad i = 1, \ldots, n, \\[2mm] \lambda^+_i(t)(u_i(t) - u^+_i) = \lambda^-_i(t)(u_i(t) - u^-_i) = 0, \quad i = 1, \ldots, n. \end{cases}$$

Since the control u is continuous, we may assume without lack of generality that there exist a nonempty subinterval J of I and an integer $m \in \{0, \ldots, n-1\}$ such that

- for $i = 1, \ldots, m$, we have $u_i(t) \in \,]u^-_i, u^+_i[$ for every $t \in J$; in this case $\lambda^+_i \equiv \lambda^-_i \equiv 0$ and

$$(S(x)^{\mathrm{T}}q)_i = \frac{\partial f}{\partial u_i}(X, u) \qquad \text{on } J;$$

- for $i = m+1, \ldots, n-1$, u_i is constant on J and equals u^-_i or u^+_i;
- $u_n \equiv 0$ vanishes on J (i.e., $i_0 = n$); as a consequence, $\lambda^+_n = \lambda^-_n = 0$ and

$$(S(x)^{\mathrm{T}}q)_n = \frac{\partial f}{\partial u_n}(X, u) \qquad \text{on } J.$$

Denote by $\bar{v} = (v_1, \ldots, v_m)$ the first m coordinates of a vector $v \in \mathbb{R}^n$. Then the minimizing control can be written as $u(t) = (\bar{u}(t), u^0)$, where $u^0 \in \mathbb{R}^{n-m}$ is constant, and

$$\overline{S(x)^{\mathrm{T}}q} = \frac{\partial f}{\partial \bar{u}}(X, u)^{\mathrm{T}} \qquad \text{on } J. \tag{26}$$

Case 1: The matrix $\frac{\partial^2 f}{\partial \bar{u}^2}(X, u)$ is invertible on a subinterval J' of J. It follows from the implicit function theorem applied to equation (26) that $\bar{u}$ is C^1 on J' and, for all $t \in J'$,

$$\dot{\bar{u}}(t) = \frac{\partial^2 f}{\partial \bar{u}^2}(X(t), u(t))^{-1}\left(\frac{d}{dt}\overline{S(x(t))^{\mathrm{T}}q(t)} - \left(L_F - \sum_i u_i(t)L_{b_i}\right)\frac{\partial f}{\partial \bar{u}}(X(t), u(t))^{\mathrm{T}}\right),$$

where L_F and L_{b_i} denote the Lie derivative with respect to F and b_i, respectively. We use (25) to eliminate $\dot{q}(t)$ in the expression

$$\frac{d}{dt}\overline{S(x(t))^{\mathrm{T}}q(t)} = \overline{DS(x(t))^{\mathrm{T}}(y)q(t)} + \overline{S(x(t))^{\mathrm{T}}\dot{q}(t)}$$

and obtain

$$\dot{u}(t) = Q_{X(t)}\left(p(t), q(t), u(t);\ \frac{\partial^2 f}{\partial \bar{u}_i \partial \bar{u}_j}, \frac{\partial^2 f}{\partial \bar{u}_i \partial X_j}, \frac{\partial f}{\partial X_i}\ \text{at}\ (X(t), u(t))\right), \qquad (27)$$

where Q_X is a rational function depending smoothly on X.

Fix now $s \in J'$. Since $\dot{X}(t) = F(X(t)) + \sum_i u_i(t)b_i(X(t))$ is never vanishing on J', we may assume, up to a local change of the coordinates $X = (X_1, \ldots, X_{2n})$ on $\mathbb{R}^{2n}$ near $X(s)$, that $F(X) + \sum_i u_i(s)b_i(X) = \frac{\partial}{\partial X_1}$. Differentiating $(S(x)^{\mathrm{T}}q)_n = \frac{\partial f}{\partial u_n}(X, u)$ with respect to time near $t = s$ leads to

$$\frac{d}{dt}(S(x(t))^{\mathrm{T}}q(t))_n = \frac{\partial^2 f}{\partial u_n \partial X_1}(X(t), u(t)) + \sum_i \Delta u_i^s(t) L_{b_i} \frac{\partial f}{\partial u_n}(X(t), u(t))$$

$$+ \sum_{i=1}^m \frac{\partial^2 f}{\partial u_n \partial \bar{u}_i}(X(t), u(t))\dot{u}_i(t),$$

where $\Delta u^s(t) = u(t) - u(s)$. We substitute the expressions of $\dot{u}(t)$ from (27) and of $\dot{q}_n$ from (25) into this equation and obtain, for t near s,

$$\frac{\partial^2 f}{\partial u_n \partial X_1} + R_X^1\left(\Delta u^s \frac{\partial^2 f}{\partial u_n \partial X_i}, \frac{\partial^2 f}{\partial u_i \partial u_j}, \frac{\partial^2 f}{\partial \bar{u}_i \partial X_j}, \frac{\partial f}{\partial \alpha_i}, p, q, u\right) = 0,$$

where R_X^1 is a rational function with coefficients depending smoothly on X, and α_i, $1 \le i \le 3n$, denotes the ith component of the vector $\alpha = (X, u)$.

Successive derivations (with substitution of $\dot{u}(t)$ by equation (27) and of $\dot{p}$ and $\dot{q}$ by equation (25) at each step) and evaluation of the derivatives at $t = s$ lead to a system of equations of the form, for $k \ge 1$,

$$\frac{\partial^{k+1} f}{\partial u_n \partial X_1^k}(X(s), u(s)) + R^k\left(P(s), \frac{\partial^j f}{\partial \alpha_{i_1} \ldots \partial \alpha_{i_j}}(X(s), u(s));\ j \le k+1\right) = 0,$$

where R^k is a rational function and if $j = k + 1$, then at least one of the α_{i_ℓ} is a $\bar{u}_i$.

Let Ω_1^N be the set of N-jets $j_{(X(s), u(s))}^N f$ such that $\det\left(\frac{\partial^2 f}{\partial \bar{u}^2}(X(s), u(s))\right) \ne 0$. It is an open subset of $\mathcal{J}_{(X(s), u(s))}^N$.

We have proved that $\left(j_{(X(s), u(s))}^N f, P(s)\right)$ belongs to $\pi_1^{-1}(0)$, where $\pi_1 \colon \Omega_1^N \times \mathbb{R}^{2n} \to \mathbb{R}^{N-1}$ is the rational mapping which takes an N-jet $j_{(X(s), u(s))}^N g \in \Omega_1^N$ and a vector $P \in \mathbb{R}^{2n}$ to

$$\left(\frac{\partial^{k+1} g}{\partial u_n \partial X_1^k}(X(s), u(s)) + R^k\left(P, \frac{\partial^j g}{\partial \alpha_1 \ldots \partial \alpha_j}(X(s), u(s));\ j \le k+1\right)\right)_{1 \le k \le N-1}.$$

This mapping is clearly surjective; therefore, $\pi_1^{-1}(0)$ is a semi-algebraic subset of $\mathcal{J}_{(X(s), u(s))}^N \times \mathbb{R}^{2n}$ of codimension $N - 1$. The projection of $\pi_1^{-1}(0)$ on $\mathcal{J}_{(X(s), u(s))}^N$ is then a semi-algebraic subset of codimension greater than $N - 2n$, which moreover contains the N-jet $j_{(X(s), u(s))}^N f$.

Case 2: The matrix $\frac{\partial^2 f}{\partial \bar{u}^2}(X, u)$ is never invertible on J. In order to show that $\bar{u}$ is C^1 and to derive an expression for $\bar{u}$, we need to introduce some notations. We define inductively a sequence of mappings $V^\ell \colon \mathbb{R}^{2n} \times \mathbb{R}^n \to \mathbb{R}^m$ as follows:

- $V^0 = \frac{\partial f}{\partial \bar{u}}^{\mathrm{T}}$;
- for a positive integer ℓ, the components of V^ℓ are

$$V_k^\ell = \begin{cases} V_k^{\ell-1} & \text{if } 1 \leq k \leq r_\ell, \\ \det\left(\dfrac{\partial V_i^{\ell-1}}{\partial \bar{u}_j}\right)_{i,j=1,\ldots,r_\ell,k} & \text{if } r_\ell + 1 \leq k \leq m, \end{cases}$$

  where $r_\ell = r_\ell(X, u)$ is the rank of the matrix $\frac{\partial V^{\ell-1}}{\partial \bar{u}}(X, u)$.

By hypothesis, $r_1(X(t), u(t))$ is smaller than m for $t \in J$. Since $X(\cdot)$ and $u(\cdot)$ are continuous, up to a permutation of the indices $\{1, \ldots, m\}$, there is a subinterval J' of J such that, for any $\ell \geq 1$,

- the rank $r_\ell(X(t), u(t))$ is constant on J';
- the function

$$\delta_\ell(X(t), u(t)) = \det\left(\frac{\partial V_i^{\ell-1}}{\partial \bar{u}_j}(X(t), u(t))\right)_{1 \leq i,j \leq r_\ell}$$

  is never vanishing on J';
- if $r_\ell < m$, then

$$V^\ell(X(t), u(t)) = \big((S(x(t))^{\mathrm{T}} q(t))_1, \ldots, (S(x(t))^{\mathrm{T}} q(t))_{r_1}, 0, \ldots, 0\big) \qquad \text{for all } t \in J'.$$

Notice that an easy induction shows the following expression:

$$V_k^\ell = \delta_1 \ldots \delta_\ell \frac{\partial^{\ell+1} f}{\partial \bar{u}_k^{\ell+1}} + G^{k,\ell}, \tag{28}$$

where $G^{k,\ell}$ is a polynomial function of the derivatives of the form $\frac{\partial^j f}{\partial \bar{u}_{i_1} \ldots \partial \bar{u}_{i_j}}$, with $j \leq \ell + 1$, each $i_l \leq k$, and $\sum_l i_l < k(\ell + 1)$.

Denote by L the largest integer such that $r_L < m$ (we set $L = +\infty$ if the latter condition is always satisfied). Then, for $\ell = 1, \ldots, L$, $V_m^\ell(X, u) \equiv 0$ on J'. If moreover $L < \infty$, then on J'

$$V^L(X, u) = \big((S(x)^{\mathrm{T}} q)_1, \ldots, (S(x)^{\mathrm{T}} q)_{r_1}, 0, \ldots, 0\big) \qquad \text{and} \qquad \frac{\partial V^L}{\partial \bar{u}}(X, u) \text{ is invertible,}$$

with $u(\cdot) = (\bar{u}(\cdot), u^0)$. It then follows from the implicit function theorem that $\bar{u}$ is C^1 on J'. Following exactly the argument of case 1, we obtain a system of equations of the form, for a fixed $s \in J'$,

$$\frac{\partial^{k+1} f}{\partial u_n \partial X_1^k}(X(s), u(s)) + R'_k = 0, \qquad k \geq 1,$$

where R'_k is a rational function of $P(s)$ and of the derivatives $\frac{\partial^j f}{\partial \alpha_{i_1} \ldots \partial \alpha_{i_j}}(X(s), u(s))$ such that $j \leq k + L$ and if one of the α_{i_ℓ} is u_n, then $j \leq k + 1$ and $j = k + 1$ implies that at least one of the other $\alpha_{i_{\ell'}}$ is a $\bar{u}_i$.

Set $M = \min(L, N - 1)$. Let Ω_2^N be the set of N-jets $j_{(X(s),u(s))}^N f$ such that

$$\delta_1(X(s), u(s)) \ldots \delta_M(X(s), u(s)) \neq 0.$$

It is thus an open subset of $\mathcal{J}_{(X(s),u(s))}^N$.

We have proved that $\left(j_{(X(s),u(s))}^N f, P(s)\right)$ belongs to $\pi_2^{-1}(0)$, where $\pi_2 \colon \Omega_2^N \times \mathbb{R}^{2n} \to \mathbb{R}^{N-1}$ is the rational mapping which takes $\left(j_{(X(s),u(s))}^N f, P(s)\right)$ to

$$\left(\left(\delta_1 \ldots \delta_\ell \frac{\partial^{\ell+1} f}{\partial \bar{u}_k^{\ell+1}}(X(s), u(s)) + G^{k,\ell}\right)_{1 \le \ell \le M}, \left(\frac{\partial^{k+1} f}{\partial u_n \partial X_1^k}(X(s), u(s)) + R_k'\right)_{1 \le k \le N-M-1}\right).$$

This mapping is clearly surjective; therefore, $\pi_2^{-1}(0)$ is a semi-algebraic subset of $\mathcal{J}_{(X(s),u(s))}^N \times \mathbb{R}^{2n}$ of codimension $N-1$. The projection of $\pi_2^{-1}(0)$ on $\mathcal{J}_{(X(s),u(s))}^N$ is then a semi-algebraic subset of codimension greater than $N-2n$, which contains the N-jet $j_{(X(s),u(s))}^N f$. $\quad\square$

Theorem 3 follows from Lemma 3 combined with standard transversality arguments. $\quad\square$

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