Hyperbolic Rigidity of Amenable Groups

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Abstract. A topological group is amenable if any of its continuous actions on a compact Hausdorff space admits an invariant probability measure. This class of groups is originated in the study Banach-Tarski paradox and has assumed an important role in topological dynamics since then. In this note, one shows that amenable groups can never admit a continuous action of general type by isometries on a geodesic separable Gromov hyperbolic space.

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1 Introduction

Let G be a topological group. It is *amenable* if whenever G acts continuously on a compact Hausdorff space K, there exists a probability measure μ on K such that it is G-invariant in the sense that $\mu(gA) = \mu(A)$ for every measurable subset $A \subseteq K$ and every $g \in G$.

A metric space (X,d) is *Gromov hyperbolic* if it satisfies Gromov four points condition (2.1). Using the Gromov products in 2.1, one can define a Gromov boundary ∂X for these spaces. If in addition, the Gromov hyperbolic space (X,d) is *proper*, *i.e.* each bounded subset is compact, then the Gromov boundary ∂X is compact [5,13].

This note mainly concerns the following folklore result: *the continuous action by isometries of an amenable group on a Gromov hyperbolic space can never be of general type*. Some special cases of this result are long known to people.

Let G be an amenable group acting continuously by isometries on a Gromov hyperbolic space (X,d). Then this action can be extended continuously to the Gromov boundary ∂X. If X is proper, then the arguments of Lemma 4.1 will show that this action cannot be of general type. When the space (X,d) is not proper, this result also holds for locally compact amenable groups, deducing from the existence of a Schottky subgroup, which will witness the non-amenability of locally compact groups [3].

In this note, we will prove the following result for amenable groups that are not necessarily locally compact under the non-proper setting:

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Theorem 1.1. *Let X be a separable geodesic space that is also δ-hyperbolic. Let G be an amenable topological group acting continuously on X by isometries. Then this cannot be of general type.*

The proof of Theorem 1.1 mainly uses the horicompactification defined as in [6]. It is sometimes also known as *metric functional compactification* [11] or *horofunction boudnary* [1], although these notions have slight nuances in its way of definition or as topological spaces. The elements in horicompactification are called *horivectors*. For a separable geodesic Gromov hyperbolic space X, we first show that each horivector is either associated to a bounded subset or a unique point on its Gromov boundary ∂X. For those who are associated to a bounded subset, we call them finite horivectors; otherwise ther are called infinite. We first prove that there is a continuous and surjective mapping from the infinite part of horicompactification to the Gromov boundary. Then we show that an $\mathrm{Isom}(X)$-invariant probability measure on the horicompactification must supported on its infinite part. Hence the push-forward probability measure will yield an $\mathrm{Isom}(X)$-invariant measure on the Gromov boundary and this goes back to the classical case of Lemma 4.1.

2 Preliminaries

2.1 Gromov hyperbolic space

Originally introduced in [9], a metric space (X,d) is *Gromov hyperbolic* or *δ-hyperbolic* for some $\delta \geq 0$ if it satisfies the *Gromov four points condition*, namely

$$\langle x,y\rangle_o \geq \min\{\langle x,z\rangle_o, \langle z,y\rangle_o\} - \delta \tag{2.1}$$

for all $x,y,z,o \in X$. Here $\langle x,y\rangle_o := \frac{1}{2}\big(d(x,o)+d(y,o)-d(x,y)\big)$ is called the *Gromov product*. Roughly speaking, the Gromov product measures the distance from the based point o to a geodesic (in fact, any geodesic) connecting x to y. If the space is δ-hyperbolic and geodesic, then we have

$$d(o,[x,y]) - \delta \leq \langle x,y\rangle_o \leq d(o,[x,y]), \tag{2.2}$$

where $[x,y]$ is any geodesic between x and y, see for example [2, III.H.1].

A sequence $(x_n)_{n\geq 0}$ in δ-hyperbolic space is *Cauchy-Gromov* if $\langle x_n, x_m\rangle_o \to \infty$ as $n,m \to \infty$. Two Cauchy-Gromov sequences $(x_n)_{n\geq 0}$ and $(y_m)_{m\geq 0}$ are *equivalent* if $\langle x_n, y_m\rangle_o \to \infty$ as $n,m \to \infty$. The *Gromov boundary* of a δ-hyperbolic space consists of all equivalent classes of Cauchy-Gromov sequences and is denoted ∂X. The Gromov product can be extended to ∂X by defining for all $x \in X$ and $\xi \in \partial X$,

$$\langle x,\xi\rangle_o := \sup\left(\liminf_{n\to\infty} \langle x, x_n\rangle_o\right),$$

where the supremum is taken among all Cauchy-Gromov sequences converging to $\xi \in \partial X$, and also by setting for all $\eta, \xi \in \partial X$,

$$\langle \eta, \xi \rangle_o := \sup \left(\liminf_{n,m\to\infty} \langle y_m, x_n \rangle_o \right)$$

in a similar way. In particular, note that $\langle \xi, \xi \rangle_o = \infty$ for all $\xi \in X \cup \partial X$. One also remarks that for any $\xi, \eta \in \partial X$ and any two sequences $x_n \to \xi$ and $y_m \to \eta$, we have

$$\langle \xi, \eta \rangle_o - 2\delta \leq \liminf_{n,m\to\infty} \langle x_n, y_m \rangle_o \leq \langle \xi, \eta \rangle_o. \tag{2.3}$$

Readers may refer to [13, §7.2, Remarque 8]. The Gromov boundary is equipped with a uniform structure induced by the Gromov product, namely generated by the basis of entourages in form of $\{(\eta, \xi) \in \partial X \times \partial X : \langle \eta, \xi \rangle_o \geq R\}$. Moreover, we can regard the *shadows*

$$\mathcal{S}_o(x, R) := \{ y \in X : \langle x, y \rangle_o \geq R \}$$

as a neighbourhood of $\xi \in \partial \mathcal{S}_o(x, R)$ in the topological space $X \cup \partial X$.

A (λ, k)-*quasi-geodesic* is a (not necessarily continuous) map $\gamma : I \to X$, where I is an interval in $\mathbb{R}$, such that for all s and t in I,

$$\frac{1}{\lambda}|s-t| - k \leq d\big(\gamma(s), \gamma(t)\big) \leq \lambda |s-t| + k.$$

A (1,0)-quasi-geodesic reduces to a geodesic and similarly, we call γ a quasi-geodesic line, ray or segment respectively if I is $\mathbb{R}$, $[0,\infty)$ or $[a,b]$.

Geometrically, a geodesic δ-hyperbolic space is where all geodesic triangles are δ-slim and if we allow geodesics to deform slightly into quasi-geodesics [2, III.H, Corollary 1.8], it will yield a very interesting result about the stability of quasi-geodesics (see [13, Chapitre 5] for example), referred as the *Morse Lemma*:

Proposition 2.1. *Let X be a δ-hyperbolic space. There exists a function $M : [1,\infty) \times [0,\infty) \to [0,\infty)$ depending on δ so that given any pair $(\lambda, k) \in [1,\infty) \times [0,\infty)$ and any two points $x, y \in X \cup \partial X$, all (λ, k)-quasi-geodesics connecting x to y are within Hausdorff distance $M(\lambda, k)$ of each other.*

Finally, let X be a Gromov hyperbolic space and $g \in \mathrm{Isom}(X)$ be an isometry. A trichotomy for such an isometry g is established by characteriing the behavior of g at infinity. To be precise, an isometry g is:

- *elliptic* if and only if $\langle g \rangle$ has bounded orbits;
- *parabolic* if and only if $d(x, g^n x)/n \to 0$ as $n \to \infty$, and in this case there is a unique fixed point $\xi \in \partial X$ by g;
- *hyperbolic* if and only if $d(x, g^n x)/n \to c > 0$ as $n \to \infty$, in this case there are a pair of points $\xi^{\pm} \in \partial X$ that is g-invariant.

If a group G acts on a Gromov hyperbolic space X by isometries, then only the following cases will happen:

- *elementary* and
  - *bounded* if it has bounded orbits;
  - *horocyclic* if it is unbounded and has no hyperbolic elements;
  - *lineal* if it has hyperbolic elements but any two hyperbolic elements have the same endpoints;
- *non-elementary* and
  - *focal* if it has hyperbolic elements, is not lineal, and any two of its hyperbolic elements have one common endpoint;
  - *general type* if it has hyperbolic elements with no common endpoint.

For these classifications, reader can find the detail in [4,5,13] under different settings.

2.2 Amenable group

A topological group is *amenable* if and only if any of its continuous actions on a compact Hausdorff space admits an invariant probability measure. There are many different but equivalent characterisation for amenable groups, see for example [8].

The amenability of a group relies on its topology. For a fixed group G, a group topology τ is an *amenable topology* for G if G becomes amenable carrying τ. Note that every group is amenable if it is equipped with the trivial topology and any coarser group topology on a given group than an amenable topology is still amenable.

A group G is said *discretely amenable* if it is so when the discrete topology is endowed. Among them are the finite groups, the abelian groups, the nilpotent groups, the solvable groups and any groups containing a finite-index subgroup of those types. So these groups remain amenable no matter what group topology they are carrying.

Amenable groups enjoy several hereditary properties. Say, every open subgroup of an amenable group is also amenable and every closed subgroup of a locally compact amenable group is also amenable. Also, the directed union of amenable groups is still amenable. Readers can refer to [8]. Moreover, it is well-knwon that the non-abelian free group $\mathbb{F}_2$ on two generators is not discretely amenable.

Now we will give some examples of amenable topological groups that are not discretely amenable.

Example 2.1. Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space and $U(\mathcal{H})$ be its unitary group. If one endows it with the strong operator topology, then it is amenable it contains a dense subgroup $U(\infty)$, which is a directed union of amenable groups $U(n)$. Meanwhile, any countable group Γ is isomorphic to a discrete, thus closed, subgroup of $U(\mathcal{H})$, in particular $\mathbb{F}_2$. Hence $U(\mathcal{H})$ is amenable with strong operator topology but not discretely amenable. See [8, Lemma 5.1, Proposition 5.2] for details.

Example 2.2. Let $\mathfrak{S}_\infty$ be the infinite symmetric group over $\mathbb{N}$. We endow it with the permutation topology, *i.e.* the topology generated by neighbourhoods of the identity element $\{\sigma \in \mathfrak{S}_\infty : \sigma(A) = A\}$ for finite subsets $A \subseteq \mathbb{N}$. Then the group becomes a Polish group. With this topology, the group $\mathfrak{S}_\infty$ is amenable since it is the directed union of amenable closed subgroups $\mathfrak{S}_n$, the symmetric group over $\{1,2,\ldots,n\}$. If we identify $\mathbb{N}$ with the elements in the free group $\mathbb{F}_2$ on two generators at the level of sets, then $\mathbb{F}_2$ can be embedded into $\mathfrak{S}_\infty$. The stabliser of the identity element $\mathrm{Id}\in\mathbb{F}_2$ is an open set in $\mathfrak{S}_\infty$ whilst the only element of $\mathbb{F}_2$ that falls into this open set is $\mathrm{Id}\in\mathbb{F}_2$ itself. So $\mathbb{F}_2$ is a non-amenable discrete (thus closed) subgroup of the amenable group $\mathfrak{S}_\infty$.

3 Compactifications of metric space

3.1 Horicompactification and others

First, consider the following mapping from the metric space X to a huge product of compact intervals

$$\begin{aligned}\Phi\colon X &\longrightarrow \prod_{y,z\in X\times X}[-d(y,z),d(y,z)]\\ x&\longmapsto \Big(d(x,y)-d(x,z)\Big)_{(y,z)\in X\times X}\end{aligned}$$

and define a compactification of X, denoted $\overline{X}^v$, by the closure of $\Phi(X)$ in the product space. An immediate application of Tychonoff's theorem implies that $\overline{X}^v$ is indeed compact. Following the usage from [6], this compactification is called the *horicompactification* of X. An element of $\overline{X}^v$ will be called a *horivector* in what follows. One remarks that $v(y,z)=v(y,x)-v(z,x)$ and $v(x,y)=-v(y,x)$ for any horivector v.

Remark 3.1. In [9, §7.5.E], such an object is given a name of "differentials of horofunctions" due to the property $v(y,z)=v(y,x)-v(z,x)$.

Another slightly different compactification of metric spaces, denoted $\overline{X}^h$, is the closure of the image of X under the following mapping

$$\begin{aligned}\Psi_o\colon X &\longrightarrow \prod_{y\in X}[-d(y,o),d(y,o)]\\ x&\longmapsto \Big(d(x,y)-d(x,o)\Big)_{y\in X}\end{aligned}$$

with a fixed base point o and it is called the *metric compactification* of X in [10, 14] for instances. The elements in $\overline{X}^h$ adopt the name of *metric functional* following [11].

One other related notion is *Busemann function* associated with a geodesic ray γ, namely $\beta_\gamma(x)=\lim_{t\to\infty}[d(x,\gamma(t))-t]$. In fact, the quantity $d(x,\gamma(t))-t$ is monotonically decreasing in $t>0$ and bounded from below by $-d(x,\gamma(0))$, see [2, II, Lemma 8.18]. It is by definition a metric functional based at $\gamma(0)$.

The following lemma shows that the horicompactification and metric compactification are essentially the same topological object and that the definition of metric compactification does not depend on the choice of the base point.

Proposition 3.1. *Let (X,d) be a metric space. For any base point $o \in X$, the two spaces $\overline{X}^v$ and $\overline{X}^h$ are homeomorphic.*

Proof. Let $A = \prod_{y\in X}[-d(y,o),d(y,o)]$ and $B = \prod_{y\in X, z\neq o}[-d(y,z),d(y,z)]$. Let π_A be the projection from the product $A\times B$ to A. Since B is compact, the projection π_A is a closed mapping (also referred as Kuratowski's theorem, see [7, Theorem 3.1.16]). Hence, we have the inclusion $\pi_A(\overline{X}^v) = \pi_A(\overline{\Phi(X)}) \subseteq \overline{\Psi_o(X)} = \overline{X}^h$ per definition of closure. Conversely, any converging net $(\Phi(x_a))_a$ in $A\times B$ yields a converging net $(\Psi_o(x_a))_a$ in A, which further implies that $\pi_A(\overline{X}^v) \supseteq \overline{X}^h$. Therefore π_A is a continuous sujection from $\overline{X}^v$ to $\overline{X}^h$. As $\overline{X}^v$ is compact and $\overline{X}^h$ is Hausdorff, in order to show that π_A is a homeomorphism, it suffices to show that π_A is injective on $\overline{X}^v$. Indeed, given two horivectors $v,w \in \overline{X}^v$ satisfying $v(x,o) = w(x,o)$ for any $x \in X$, we have

$$v(y,z) - w(y,z) = v(y,o) - v(z,o) - w(y,o) + w(z,o) = 0$$

for any $y,z \in X$. □

The following observation is essential for continuous group actions by isometries on metric spaces:

Proposition 3.2. *Let (X,d) be a metric space and G be any topological group acting on X continuously by isometries. Then the action of G on $\overline{X}^v$ is continuous,* i.e. *$\overline{X}^v$ is a G-flow.*

Proof. As horivectors are 1-Lipschitz in both variables, it soon yields that the mapping Φ is continuous and it turns out that $\overline{X}^v$ is an $\mathrm{Isom}(X)$-flow for the pointwise convergence topology of $\mathrm{Isom}(X)$ ([6, Lemma 2.5]). Now let G acts continuously by isometries on (X,d). Then there is a continuous homomorphism $G \to \mathrm{Isom}(X)$, which further makes $\overline{X}^v$ a G-flow. □

horofunctions or Busemann functions have served as replacement for linear functionals when the space is not linear. In a recent paper [11], Karlsson establishes a similar result for metric functionals. Thanks to the homeomorphism that we establish in Proposition 3.1, we can reformulate his result in terms of horivectors without difficulty.

Proposition 3.3 (Hahn-Banach for horivectors)**.** *Let (Y,d) be a metric space and X be a subspace of Y. Then for every horivector $v \in \overline{X}^v$, there exists a horivector $V \in \overline{Y}^v$ that extends v in the sense that $V(y,z) = v(y,z)$ for all $y,z \in X$.*

3.2 Busemann sequences

Now we will adapt some results from [12] in terms of horivectors.

In what follows, we will refer to $O(\delta)$ an additive error at most a multiple of δ, *viz.* if we write $f(x) = g(x) + O(\delta)$, then it means $|f(x) - g(x)| < M\delta$ for some uniform $M > 0$ independent of x. Errors $O(\delta)$ appear in different places can be different.

Let us introduce the notion of *orientation* of a geodesic γ, which is a strict total order on the points of γ induced by an isometric parameter $\gamma^+ : I \to X$. Suppose that $x = \gamma^+(t)$, $y = \gamma^+(s)$ and we say that $x \geq y$ if and only if $t \geq s$. Then given a fixed orientation, the *signed distance function* on a geodesic is defined by

$$d_\gamma^+(x,y) = \begin{cases} d(x,y), & \text{if } x \leq y \\ -d(x,y), & \text{if } x \geq y \end{cases}.$$

Proposition 3.4. *Let X be a δ-hyperbolic space and γ be a geodesic in X. For any horivector $v \in \overline{X}^v$, up to an additive error at most a multiple of δ and independent of γ, either there exists a $p \in \gamma$ such that*

$$v(y,z) = v(p,z) + d(y,p) + O(\delta), \quad (\forall y \in \gamma), \tag{3.1}$$

or there is an orientation for γ so that for every $p \in \gamma$

$$v(y,z) = v(p,z) + d_\gamma^+(y,p) + O(\delta), \quad (\forall y \in \gamma). \tag{3.2}$$

Proof. It follows from [12, Proposition 3.6]. □

Remark 3.2. From the proof we can see that, whenever the concerned geodesic γ is a segment, the case (3.1) will always hold. The case (3.2) can happen only when a Busemann sequence is fellow-travelling with a geodesic ray, and in that case this sequence will be Cauchy-Gromov.

For a metric space (X,d), we say that a sequence $(x_n)_{n\geq 0}$ is *Busemann at base point o* if $\Psi_o(x_n)$ converges to a metric functional in $\overline{X}^h$; it is *Busemann* if $\Phi(x_n)$ converges to a horivector in $\overline{X}^v$. Note that if a sequence is Busmann, then it is Busmann at any point $o \in X$.

Busemann sequences can be used to give a classification for horivectors, as well as metric functionals, on a δ-hyperbolic space. The first type of Busemann sequence is Cauchy-Gromov.

Lemma 3.1. *Let X be a δ-hyperbolic space and $(x_n)_{n\geq 0}$ be a Busemann sequence in it. Assume that $(x_n)_{n\geq 0}$ has a Cauchy-Gromov subsequence, then it is itself also Cauchy-Gromov.*

Proof. Suppose that it has a subsequence $(x_{n_k})_{k\geq 0}$ converging to $\xi \in \partial X$. By assumption, for any $K > 0$, there exists a $x \in X$ with $d(x,o)$ much larger than $2K$ such that $x_{n_k} \in \mathcal{S}_o(x,2K)$ for large enough k, *i.e.* for any $M > 0$, there exists $m > M$ such that $\langle x_m, x\rangle_o \geq 2K$. Since the sequence is Busemann, there exists $N > 0$ such that for all $n,m > N$

$$|\Phi(x_n)(x,o) - \Phi(x_m)(x,o)| = 2|\langle x_n, x\rangle_o - \langle x_m, x\rangle_o| < K.$$

In particular, by taking a particular $x_m \in \mathcal{S}_o(x,2K)$, we can deduce that $\langle x_n, x\rangle_o \geq K$ and that by consequence $x_n \in \mathcal{S}_o(x,K)$ for all $n \geq N$. This implies that x_n converge to $\xi \in \partial X$ as a Cauchy-Gromov sequence. □

The following lemma will finish establishing a dichotomy for Busemann sequences and also horivectors in a δ-hyperbolic space.

Lemma 3.2. *Let X be a δ-hyperbolic space and $(x_n)_{n\geq 0}$ be a Busemann sequence in it. Suppose in addition that $(x_n)_{n\geq 0}$ converges to a horivector $v\in\overline{X}^v$ and is not Cauchy-Gromov, then the function $v(\cdot,z)$ is bounded from below for any $z\in X$.*

Proof. Fix any $z\in X$. Suppose for contradiction that $v(\cdot,z)$ is not bounded from below. Then for any $M>0$, there exists an $y\in X$ such that $v(y,z)<-2M$. In terms of limit, it means that there is an $N>0$ such that $d(x_n,y)-d(x_n,z)<-M$ for all $n\geq N$. So for any $n,m\geq N$, we will have

$$\begin{aligned}\langle x_n,x_m\rangle_z &= \frac{1}{2}\left(d(x_n,z)+d(x_m,z)-d(x_n,x_m)\right)\\ &> \frac{1}{2}\left(d(x_n,y)+d(x_m,y)+2M-d(x_n,x_m)\right)>M,\end{aligned}$$

which implies that $(x_n)_{n\geq 0}$ must be Cauchy-Gromov. Contradiction! □

3.3 Horiboundary

For horicompactification, we can define the *horiboundary* by $\overline{X}^v\setminus\Phi(X)$, denoted $\partial\overline{X}^v$.

Previously we give a dichotomy for Busemann sequences in a δ-hyperbolic space. The same dichotomy for horivectors can be established by passing to limits.

Let $\overline{X}^v_\infty$ be the subset of $\partial\overline{X}^v$ so that any of its elements is not not bounded from below. Also denote by $\overline{X}^v_f$ the set of horivectors v such that $v(\cdot,z)$ is bounded below for any $z\in X$. We note that $\overline{X}^v=\overline{X}^v_\infty\cup\overline{X}^v_f$ and $\overline{X}^v_\infty\subseteq\partial\overline{X}^v$.

On one hand, each horivector in $\overline{X}^v_f$ is uniquely corresponded to a bounded part in X. Let $v\in\overline{X}^v_f$. Define the *coarse minima based at z of v* by

$$\mathcal{L}(v,z):=\left\{y\in X: v(y,z)\leq\inf_{x\in X}v(x,z)+1\right\}.$$

Similarly, its *coarse minima* $\mathcal{L}(v)$ is then defined as the union of $\mathcal{L}(v,z)$ for all $z\in X$. The same arguments for [12, Lemma 3.13] will yield the following result:

Proposition 3.5. *Let X be a geodesic δ-hyperbolic space and $v\in\partial\overline{X}^v_f$. Then there exists a constant K only depending on δ such that* $\operatorname{diam}\left(\mathcal{L}(v)\right)\leq K$.

On the other hand, using *minimising sequence* for such horivector $v\in\overline{X}^v_\infty$, *i.e.* a sequence $(y_n)_{n\geq 1}$ such that $v(y_n,z)\to-\infty$ as $n\to\infty$, one can construct a *boundary correspondence* $\Xi\colon\overline{X}^v_\infty\to\partial X$ between the infinite part of horboundary and Gromov boundary [12, Lemma 3.10].

Proposition 3.6 ([12]). *Let X be a separable geodesic Gromov hyperbolic space. Then the boundary correspondence $\Xi\colon\overline{X}^v_\infty\to\partial X$ is continuous and surjective.*

4 Dynamics of amenable groups

Let X be a separable geodesic δ-hyperbolic space and D be a dense subset in X. Then the projection
$$\overline{X}^v \to \prod_{x,y\in D}[-d(x,y),d(x,y)]$$
is a homeomorphism. Hence we need only to treat the valuation of horivectors on the dense subset D with base point $o\in D$. As a result, the expression
$$\overline{X}^v_\infty = \bigcap_{N>0}\bigcup_{x\in D}\left\{v\in\overline{X}^v : v(x,o) < -N\right\}$$
implies that $\overline{X}^v_\infty$ and $\overline{X}^v_f = \overline{X}^v \setminus \overline{X}^v_\infty$ are Borel sets in $\overline{X}^v$.

Suppose that G is an amenable group acting continuously by isometries on X. Proposition 3.2 asserts that the action of G on the horicompactification $\overline{X}^v$ is continuous. Hence there exists a G-invariant probability measure μ on $\overline{X}^v$.

Proposition 4.1. *Let X be a separable geodesic δ-hyperbolic space. Suppose that G is a group acting continuously by isometries on X and suppose that the action is unbounded. Let μ be a G-invariant probability measure on $\overline{X}^v$. Then $\mu(\overline{X}^v_f)=0$.*

Proof. Let D be a dense subset in X and $o\in D$ be a base point. For a fixed $q\in\mathbb{Q}$ and two distinct points $x,y\in D$, the open set
$$V(x,y,q) = \overline{X}^v \cap \left([-d(x,o),q)\times(q,d(y,o)]\times\prod_{(z,w)\neq(x,o),(y,o)}[-d(z,w),d(z,w)]\right)$$
is the collection of all horivectors v in $\overline{X}^v$ such that $v(x,o)<q<v(y,o)$. Therefore, the union $V_{x,y}=\bigcup_{q\in\mathbb{Q}}V(x,y,q)$ is the collection of the horivectors $v\in\overline{X}^v$ such that $v(x,o)<v(y,o)$.

Let $R>0$ be a positive real number that is larger than the uniform diameter K of coarse minima $\mathcal{L}(v)$ from Proposition 3.5. For any $z\in D$, we define
$$Y(z,R) = \left\{v\in\overline{X}^v_f : \mathcal{L}(v)\cap B(z,R)\neq\emptyset\right\}$$
where $B(z,R)=\{p\in X : d(z,p)<R\}$. One remarks that $\mathcal{L}(v)$ contains $\mathcal{L}(v,o)$. So if $v\in Y(z,R)$, then $\mathcal{L}(v,o)$ will be contained in $B(z,2R)$. By consequence, for every $y\notin B(z,2R)$, we will have
$$v(y,o) > \inf_{x\in X} v(x,o)+1 > \inf_{x\in X} v(x,o) = \inf_{x\in B(z,2R)} v(x,o).$$
By the density of D in X, it turns out that there must be some $x\in B(z,2R)\cap D$ such that $v(x,o)<v(y,o)$. So if one sets $B(z,2R)^c$ to be the complement of $B(z,R)$ in X and
$$Y'(z,R) = \left\{v\in\overline{X}^v_f : \forall y\in B(z,2R)^c\cap D\,\exists x\in B(z,2R)\cap D \text{ so that } v(x,o)<v(y,o)\right\},$$

then $Y(z,R) \subseteq Y'(z,R)$. In fact, the set $Y'(z,R)$ remains the same even if one removes the constraint of x,y being in D from the definition. But the definition above means that we have the following expression

$$Y'(z,R) = \bigcap_{y \in B(z,2R)^c \cap D} \left(\bigcup_{x \in B(z,2R) \cap D} V_{x,y} \right).$$

This shows that $Y'(z,R)$ is a Borel set in $\overline{X}^v$.

Let $g \in G$ be any isometry. For any $v \in Y'(z,R)$, we can see that for any $g^{-1}y \notin B(g^{-1}z,2R)$, there exists a $g^{-1}x \in B(g^{-1}z,2R)$ such that

$$\begin{aligned}(gv)(y,o) &= v(g^{-1}y,g^{-1}o)\\ &= v(g^{-1}y,o) + v(o,g^{-1}o)\\ &> v(g^{-1}x,o) + v(o,g^{-1}o)\\ &= v(g^{-1}x,g^{-1}o)\\ &= (gv)(x,o),\end{aligned}$$

which implies that $gv \in Y'(g^{-1}z,R)$. Hence $gY'(z,R) \subseteq Y'(g^{-1}z,R)$. If we apply the same argument for g^{-1} and $Y'(g^{-1}z,R)$, then the inclusion above is actually equality, *i.e.* $gY'(z,R) = Y'(g^{-1}z,R)$.

Now we claim that $\mu(Y'(z,R)) = 0$ for all $z \in X$. Indeed, suppose *ab absurdo* that there is a point $z \in X$ such that $\mu(Y'(z,R)) > 0$. Note that if $v \in Y'(z,R)$, then $\mathcal{L}(v,o) \cap B(z,2R) \neq \emptyset$, which further implies that $\mathcal{L}(v) \subset B(z,3R)$. So if $d(z_1,z_2) > 6R$, then $Y'(z_1,R) \cap Y'(z_2,R) = \emptyset$. As the action of G on X is unbounded, by picking a sequence $(g_n)_{n\in\mathbb{N}}$ of elements in G such that $d(g_n z, g_m z) > 6R$ for every $n \neq m$, we will have

$$1 = \mu\left(\overline{X}^v\right) \geq \mu\left(\bigcup_{n\in\mathbb{N}} Y'(g_n z,R)\right) = \sum_{n\in\mathbb{N}} \mu(Y'(g_n z,R)) = \sum_{n\in\mathbb{N}} \mu(Y'(z,R)) = \infty,$$

which is not possible!

Finally, by density of D in X, we have $\overline{X}^v_f \subseteq \bigcup_{z\in D} Y(z,R)$. So we get

$$0 \leq \mu\left(\overline{X}^v_f\right) \leq \mu\left(\bigcup_{z\in D} Y(z,R)\right) \leq \sum_{z\in D} \mu(Y(z,R)) \leq \sum_{z\in D} \mu(Y'(z,R)) = 0.$$

This completes the proof. □

Now we shall prove the following general result:

Lemma 4.1. *Let X be a δ-hyperbolic space and let G act on X by isometries. Suppose that there exists a G-invariant probability measure ν on ∂X. Then the action of G cannot be of general type.*

Proof. Suppose for contradiction that the action of G on X is of general type. Then G has no finite orbit on ∂X. Because ν is G-invariant, so ν is atomless, otherwise the total measure of ∂X under ν will exceed 1. Moreover, there exists at least one hyperbolic isometry g in G. Let $\partial_X\langle g\rangle=\{\xi_\pm\}$. Since $\xi_\pm$ are not atoms, we can find open neighbourhoods $U_\pm$ of $\xi_\pm$ respectively such that $\nu(U_\pm)<1/3$. This means that $\nu(\partial X\setminus U_-)>2/3$. Yet by North-South dynamic of g (see [5, §6.1]), there exists $n>0$ such that $g^n(\partial X\setminus U_-)$ is contained in U_+. Hence we have

$$\frac{2}{3}<\nu(\partial X\setminus U_-)=\nu(g^n(\partial X\setminus U_-))\leq\nu(U_+)<\frac{1}{3}.$$

This is absurd! □

Now we are able to prove Theorem 1.1:

Proof of Theorem 1.1. Suppose that G is an amenable group act continuously by isometries on a separable geodesic Gromov hyperbolic space (X,d). Then there is a G-invariant probability measure on $\overline{X}^v$, denoted μ. By Proposition 4.1, the probability measure μ must supported on $\overline{X}^v_\infty$. The continuous boundary correspondence $\Xi\colon\overline{X}^v_\infty\to\partial X$ from Proposition 3.6 will yield a G-invariant probability measure $\Xi_*(\mu)$ by pushing-forward. Now the result comes immediately from Lemma 4.1. □

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