INVERSE PROBLEMS

M2 AI — SIGNAL PROCESSING

DIRECT PROBLEM

- Let
  - $x \in \mathbb{R}^N$ be the signal of interest (clean image, music, etc.)
  - $A \in \mathbb{R}^{MN}$ be a known linear operator (sensing matrix, mixing matrix, diffusion matrix, etc.)
  - $y \in \mathbb{R}^M$ be the (noisy) observed/measured signal
  - $e \in \mathbb{R}^M$ be some noise (assumed to be white Gaussian noise)
- The direct problem is :

$$y = Ax + n$$

INVERSE PROBLEM

- The goal of the inverse problem is to estimate the original signal x from the measurement

$$y = Ax + n$$

- If $M \geq N$ the problem is said (over)-determined
- If $M < N$ the problem is said under-determined

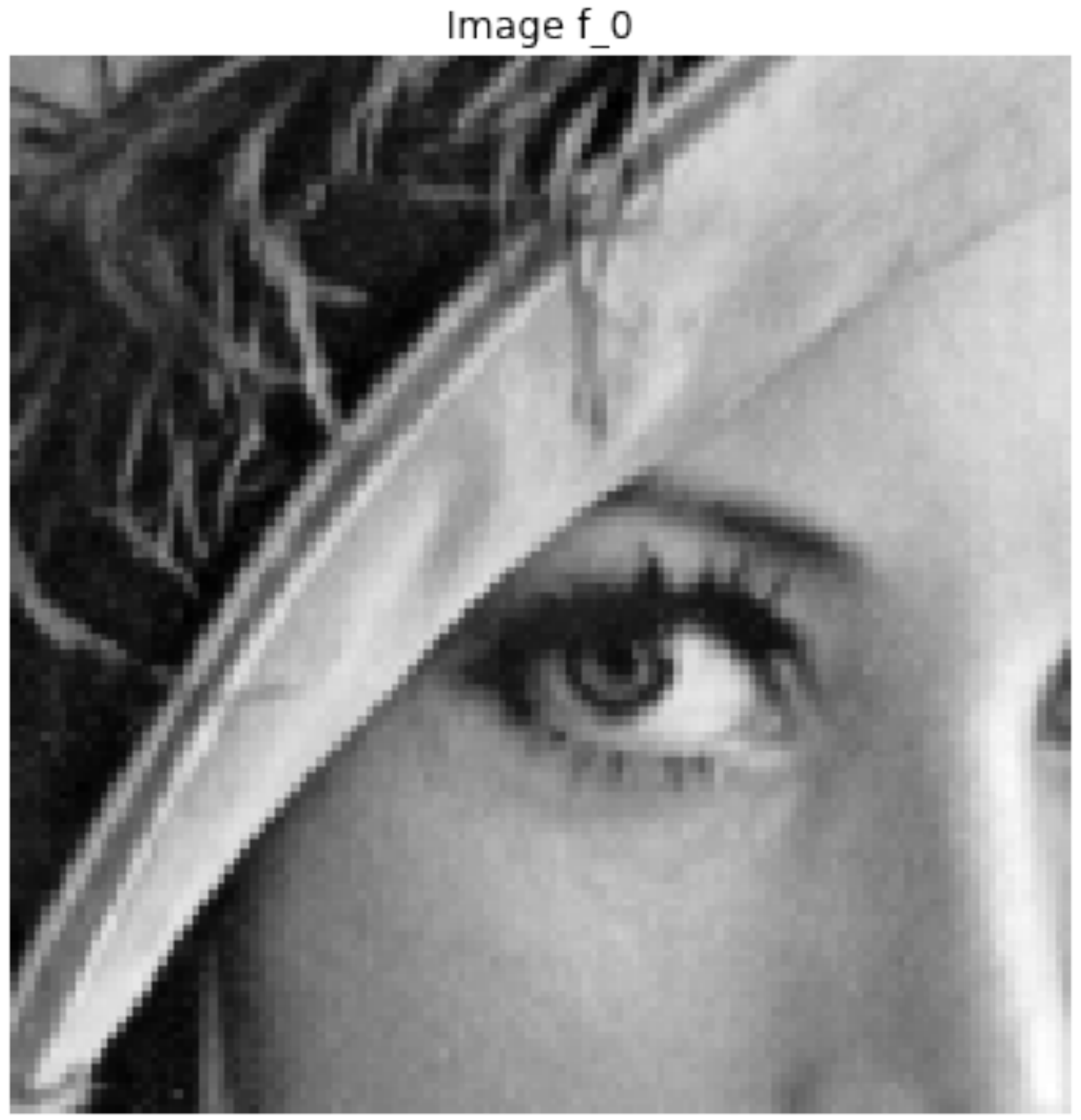
INVERSE PROBLEM: EXAMPLES

- Denoising: $y = x + n$

- Deconvolution: $y = h * x + n = Hx + n$ or $\hat{y} = \hat{h} \odot \hat{x} + \hat{n}$

INVERSE PROBLEM: EXAMPLES

- Compressive sensing: A is a random matrix with $M < N$
- Inpaiting: A is a binary mask



OPTIMIZATION FRAMEWORK

- We seek an estimation of x by

$$x = \text{argmin}_x \ \mathscr{L}(y, A, x) + \lambda \mathscr{R}(x)$$

- where
  - $\mathscr{L}(y, A, x)$is the loss: models the link between the signal x and the observation y through the operator A
  - $\mathscr{R}(x)$is the loss: models the prior on the signal x
  - $\lambda > 0$ is some hyper-parameter

OPTIMIZATION FRAMEWORK

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$$x = \operatorname{argmin}_x \mathscr{L}(y, A, x) + \lambda \mathscr{R}(x)$$

- Loss:
  - $\frac{1}{2}\|y - Ax\|_2^2$: energy of the residual, adapted to white Gaussian noise
  - $\|y - Ax\|_1$: robust regression
- Regularization:
  - $\frac{1}{2}\|x\|_2^2$: energy of the signal
  - $\frac{1}{2}\|\nabla x\|_2^2$: energy of the derivative
  - $\|x\|_1$: sparsity of the signal
  - $\|\nabla x\|_1$: sparsity of the derivative (total variation)

USE OF A DICTIONARY

- $\mathscr{R}(x)$ can be difficult to chose
- Idea: use a dictionary (such as Wavelets or time-frequency), where the signal is known to be sparse (well represented by few coefficients)
- Let $\Phi \in \mathbb{R}^{NK}$ be such a dictionary, with $x = \Phi\alpha$. α are called the synthesis coefficients
- The direct problem writes

$$y = Ax + n = A\Phi\alpha + n$$

- The minimization becomes

$$\alpha = \text{argmin} \frac{1}{2} \|y - A\Phi\alpha\|_2^2 + \lambda \|\alpha\|_1$$

- and $x = \Phi\alpha$

INVERSE PROBLEM: ALGORITHM

- How to minimize

$$\alpha = \text{argmin} \frac{1}{2} \|y - A\Phi\alpha\|_2^2 + \lambda \|\alpha\|_1$$

  - It is a non-smooth convex problem
  - Known as the Lasso or Basis-Pursuit Denoising
- Consider the "simple" denoising problem (e.g. Φ is orthogonal)

$$\alpha = \text{argmin} \frac{1}{2} \|y - \alpha\|_2^2 + \lambda \|\alpha\|_1$$

- We can show that the solution is given by the so-called Soft-Thresholding operator :

$$\alpha = \mathcal{S}_\lambda\left(y\right) = y \left(1 - \frac{\lambda}{|y|}\right)^+$$

INVERSE PROBLEM: ALGORITHM

- How to minimize

$$\alpha = \text{argmin} \frac{1}{2} \|y - A\Phi\alpha\|_2^2 + \lambda \|\alpha\|_1$$

- FISTA (Fast Iterative Shrinkage/Thresholding Algorithm)
  - Initialization: $\alpha^{(0)} \in \mathbb{R}^N$, $z^{(0)} \in \mathbb{R}^N$, $L \leq ||A\Phi||^2$, $t = 0$
  - Do until convergence :
    - $\alpha^{(t+1)} = \mathcal{S}_{\lambda/L} \left(z^{(t)} + \frac{1}{L} \Phi^* A^* \left(y - A\Phi z^{(t)} \right) \right)$
    - $z^{(t+1)} = \alpha^{(t+1)} + \frac{t}{t+5} \left(\alpha^{(t+1)} - \alpha^{(t)} \right)$
  - Remark: we have $\|A\Phi\|^2 \leq \|A\|^2 \|\Phi\|^2$, then if Φ is a Parseval frame, we simply have $L \leq \|A\|^2$.

FISTA WITH WARM RESTART

- In practice, the algorithm must be run with various values of lambda.
- When $\lambda \to 0$ we have $||y - A\Phi\alpha|| \to 0$.
- When $\lambda = ||\Phi^* A^* y||_\infty$ the solution is $\alpha = 0$
- One can choose these values distributed on a log scale $\in [10^{-4}||\Phi^* A^* y||_\infty, ||\Phi^* A^* y||_\infty]$, with a fixed number of λ, such as we have $\lambda_1 < \lambda_2 < \ldots < \lambda_I$
- The idea is to initialize the algorithm for λ_i with the results get from λ_{i-1}

FISTA WITH THRESHOLDING RULES

- The Soft-thresholding can be replaced by any thresholding rules
- Some examples:
  - Hard Thresholding: $\mathscr{H}_\lambda(\alpha) = \alpha \mathbf{1}_{|\alpha|<;\lambda}$
  - Empirical Wiener: $\mathscr{S}_\lambda^{EW}(\alpha) = \alpha \left(1 - \frac{\lambda^2}{|\alpha|^2} \right)^+$

TO DO (1/2): INPAINTING (DEADLINE: 31/12)

- Data
  - Image or signal you want
- Todo
  - Simulate various inpainting problems as follows
    - Generate a random binary matrix A of the same size as the signal, with a parameter p controlling the Bernoulli law
    - Add some white Gaussian noise (at various levels)
    - Generate the direct problem $y = Ax + b$ (where $x$is the original signal)
    - Estimate x using the sparse approach (reminder: an audio signal (resp. image) is sparse in the time-frequency (resp. Wavelet) domain)
  - Discuss the results obtained by changing:
    - the sparse representation (various wavelets, various STFT parameters…)
    - the thresholding rules (soft, hard, empirical Wiener)
    - the choice of the λ parameter
  - Discussion should be made concerning the value of p and the level of noise

TO DO (2/2): COMPRESS SENSING (DEADLINE: 31/12)

- Data
  - Image or signal you want of size N
- Todo
  - Simulate various compress sensing problems as follows
    - Generate a random normal matrix A of size MxN, M < N, with a given mean and a given covariance matrix (start with a standard normal pdf)
    - Add some white Gaussian noise (at various levels)
    - Generate the direct problem $y = Ax + b$ (where x is the original signal)
    - Estimate x using the sparse approach (reminder: an audio signal (resp. image) is sparse in the time-frequency (resp. Wavelet) domain)
  - Discuss the results obtained by changing:
    - the sparse representation (various wavelets, various STFT parameters…)
    - the thresholding rules (soft, hard, empirical Wiener)
    - the choice of the λ parameter
  - Discussion should be made concerning the size of A, its mean and its covariance matrix, as well as the level of noise.