SPARSE RECOVERY

SPARSE REPRESENTATIONS

SPARSE RECOVERY

PROBLEM

- Let $x \in \mathbb{R}^N$ be a $s-$sparse vector, ie such that $\|x\|_0 = s$
- Let $A \in \mathbb{R}^{MN}$ be a measurement matrix, and let

$$y = Ax$$

- When can x be exactly recovered from y ?

All the presented results and much more can be found in:
A Mathematical Introduction to Compressive Sensing, S. Foucart & H. Rauhut

REMINDER: ALGORITHMS

- Matching Pursuit and Orthogonal Matching Pursuit
- Basis Pursuit
- Iterative Thresholding

NULL SPACE PROPERTY

- Let $A \in \mathbb{R}^{MN}$ be a matrix with normalized columns ($\|a_i\|_2 = 1 \ \forall i$).
- A satisfies the null space property (NSP) relative to a set S iff

$$\|v_S\|_1 < \|v_{\bar{S}}\|_1 \text{ for all } v \in \ker(A) \backslash \{0\}$$

- Equivalently, $A \in \mathbb{R}^{MN}$satisfies the NSP relative to a set S iff

$$2\|v_S\|_1 \leq \|v\|_1 \text{ for all } v \in \ker(A) \backslash \{0\}$$

- A satisfies the null space property of order s if ti satisfies the NSP for all S such that card$(S) \leq s$

NULL SPACE AND S-SPARSE RECOVERY

- Given a matrix $A \in \mathbb{R}^{MN}$, every vector $x \in \mathbb{R}^N$ such that $\mathrm{supp}(x) = S$ is the unique solution of $y = Ax$ iff A satisfies the NSP relative to S
- Given a matrix $A \in \mathbb{R}^{MN}$, every vector $s-$sparse vector $x \in \mathbb{R}^N$ is the unique solution of $y = Ax$ iff A satisfies the NSP of order s

PROOF

- Suppose that for all x supported on S, x is the unique minimizer of $\|z\|_1 \ s.t \ Ax = Az$.

Let $v \in \ker(A) \backslash \{0\}$, v_S is then the unique minimizer of $\|z\|_1 \ s.t \ Av_S = Az$.
Moreover, we have $0 = Av = A(v_{\bar{S}} + v_S)$, then $A(-v_{\bar{S}}) = Av_S$ and necessarily we have $\|v_S\|_1 < \|v_{\bar{S}}\|_1$ by unicity of v_S. Then A satisfies the NSP relative to S

- Suppose that A satisfies the NSP relative to S. Let x supported on S and a vector $z \neq x$ such that $Ax = Az$. Then $v = x - z \in \mathrm{Ker}(A) \backslash \{0\}$. Then

$\|x\|_1 = \|x - z_S + z_S\|_1 \leq \|x - z_S\|_1 + \|z_S\|_1 = \|v_S\|_1 + \|z_S\|_1 < \|v_{\bar{S}}\|_1 + \|z_S\|_1$ (because of the NSP)
$\|x\|_1 < \|z_{\bar{S}}\|_1 + \|z_S\|_1 = \|z\|_1$, then x minimizes $\|x\|_1$

COHERENCE: DEFINITION

- Let $A \in \mathbb{R}^{MN}$ be a matrix with normalized columns ($\|a_i\|_2 = 1 \ \forall i$).
- The coherence $\mu = \mu(A)$ of the matrix A is given by

$$\mu = \max_{i \neq j} |\langle a_i, a_j \rangle|$$

- Remark: $\mu \leq 1$

COHERENCE: PROPERTIES

- Let $A \in \mathbb{R}^{MN}$ be a matrix with normalized columns. Then

$$\mu \geq \sqrt{\frac{N-M}{M(N-1)}}$$

- The equality holds iff A is an equi-angular tight frame (ie $\langle a_i, a_j \rangle = Cte \ (\forall i \neq j)$)

COHERENCE AND SPARSE RECOVERY

- Let $A \in \mathbb{R}^{MN}$ be a matrix with normalized columns. Then, every $s-$sparse vectors x can be recovered from $y = Ax$ by the basis pursuit and the matching pursuit algorithms if

$$\mu < \frac{1}{2s - 1}$$

BABEL FUNCTION: DEFINITION

- Let $A \in \mathbb{R}^{MN}$ be a matrix with normalized columns ($\|a_i\|_2 = 1 \ \forall i$).
- The babel-1 function, or ℓ_1-coherence $\mu_1(s)$ of the matrix A is given by

$$\mu_1(s) = \max_i \max \left\{ \sum_{j \in S} |\langle a_i, a_j \rangle| \, , \text{card}(S) = s, i \notin S \right\}$$

- Remark: $\mu \leq \mu_1(s) \leq s\mu$

BABEL-1 FUNCTION AND SPARSE RECOVERY

- Let $A \in \mathbb{R}^{MN}$ be a matrix with normalized columns. Then, every $s-$sparse vectors x can be recovered from $y = Ax$ by the basis pursuit and the matching pursuit algorithms if

$$\mu_1(s) + \mu_1(s-1) < 1$$

PROOF FOR BP

Let $v \in \mathrm{Ker}(A)\backslash\{0\}$. Then $Av = 0 \Leftrightarrow \sum_i a_i v_i = 0$. Let a set S such that $\mathrm{card}(S) = s$ and let $k \in S$. Then $\langle a_k, Av\rangle = \sum_i v_i \langle a_k, a_i\rangle = 0$, then $v_k = v_k\langle a_k, a_k\rangle = -\sum_{i\neq k} v_i\langle a_i, a_k\rangle = -\sum_{i\in\bar{S}} v_i\langle a_i, a_k\rangle - \sum_{i\in S, i\neq k} v_i\langle a_i, a_k\rangle$

Consequently, we have $|v_k| \leq \sum_{i\in\bar{S}} |v_i||\langle a_i, a_k\rangle| + \sum_{i\in S, i\neq k} |v_i||\langle a_i, a_k\rangle|$

Finally $\|v\|_1 = \sum_k |v_k| < \sum_{i\in\bar{S}} |v_i| \sum_k |\langle a_i, a_k\rangle| + \sum_k \sum_{i\in S, i\neq k} |v_i| \sum_k |\langle a_i, a_k\rangle| < \|v_{\bar{S}}\|_1 \mu_1(s) + \|v_S\|_1 \mu_1(s-1)$

Which leads to

$\|v_S\|_1 < \dfrac{1-\mu_1(s-1)}{\mu_1(s)}\|v_S\|_1 < \|v_{\bar{S}}\|_1$, i.e. A satisfies the NSP

BABEL-1 FUNCTION AND SPARSE RECOVERY

- Let $A \in \mathbb{R}^{MN}$ be a matrix with normalized columns. Then, every $s-$sparse vectors x can be recovered from $y = Ax$ by one step of Hard Thresholding if

$$\mu_1(s) + \mu_1(s-1) < \frac{\min_{i \in S} |x_i|}{\max_{i \in S} |x_i|}$$

Where $S = \text{supp}(x)$

PROOF

- We have to show that $\forall k \in S, \forall j \in \bar{S}$ $|\langle a_k, y\rangle| > |\langle a_j, y\rangle|$, ie $|\langle Ax, a_k\rangle| > |\langle Ax, a_j\rangle|$
- $|\langle Ax, a_j\rangle| = |\sum_{i\in S} x_i \langle a_i, a_j\rangle| \leq \sum_{i\in S} |x_i||\langle a_i, a_j\rangle| \leq \mu_1(s)\max_{i\in S}|x_i|$
- $|\langle Ax, a_k\rangle| = |\sum_{i\in S} x_i \langle a_i, a_k\rangle| \geq |x_j| - \sum_{i\in S, i\neq j} |x_i||\langle a_i, a_k\rangle| \geq \min_{i\in S}|x_i| - \mu_1(s-1)\max_{i\in S}|x_i|$
- Then $|\langle Ax, a_k\rangle| - |\langle Ax, a_j\rangle| \geq \min_{i\in S}|x_i| - (\mu_1(s) + \mu_1(s-1))\max_{i\in S}|x_i| > 0$

BABEL-1 FUNCTION AND SPARSE RECOVERY

- Let $A \in \mathbb{R}^{MN}$ be a matrix with normalized columns. Then, every $s-$sparse vectors x can be recovered from $y = Ax$ by at most s iterations of the iterated Hard Thresholding if

$$2\mu_1(s) + \mu_1(s-1) < 1$$

Where $S = \text{supp}(x)$

- IHT:

$$x^{(t+1)} = \mathscr{H}_s\left(x^{(t)} + A^*(y - Ax^{(t)})\right)$$

Where $\mathscr{H}_s(x)$ keeps the s largest magnitude value of x

RESTRICTED ISOMETRY CONSTANT (RIC)

- The sth restricted isometry constant δ_s of a matrix $A \in \mathbb{R}^{MN}$ is the smallest $\delta \geq 0$ such that

$$(1-\delta)\|x\|^2 \leq \|Ax\|^2 \leq (1+\delta)\|x\|^2$$

For all $s-$sparse vector $x \in \mathbb{R}^N$

- Equivalently

$$\delta_s = \max_{S, \text{card}(S) \leq s} \|A_s^* A_s - Id\|$$

RIC

- The RIC δ_s are increasing
- If A has normalized columns, a coherence μ and a Babel-1 function μ_1, then

$$\delta_1 = 0, \delta_2 = \mu, \delta_s \leq \mu_1(s-1)$$

RESTRICTED ISOMETRY PROPERTY (RIP) AND BP

- Let A such that $\delta_{2s} \leq \frac{1}{3}$
- Then every $s-$sparse vector x is the unique solution of

$$\min_z \|z\|_1 \quad \text{s.t.} \quad Ax = Az$$

RESTRICTED ISOMETRY PROPERTY (RIP) AND IHT

- Let A such that $\delta_{3s} \leq \frac{1}{2}$
- Then, for every $s-$sparse vector x such that $y = Ax$, the IHT initialized by a $s-$ sparse vector converges to x
- IHT:

$$x^{(t+1)} = \mathscr{H}_s\left(x^{(t)} + A^*(y - Ax^{(t)})\right)$$

Where $\mathscr{H}_s(x)$ keeps the s largest magnitude value of x

COHERENCE OR RIP ?

- Coherence: easy to check, but very restrictive (few matrix satisfies coherence properties)
- RIP: difficult to check. Satisfied with some random matrices
- Application: compressive sensing