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THE FORMATION OF COOPER PAIRS  
AND THE NATURE OF SUPERCONDUCTING CURRENTS

Victor F. Weisskopf

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AND THE NATURE OF SUPERCONDUCTING CURRENTS

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ABSTRACT

A simple physical explanation is given for the formation of Cooper pairs in a superconducting metal, for the origin of the attractive force causing the binding of the pairs, for the forming of a degenerate Bose gas by the Cooper pairs, for the finite energy gap that prevents the ensemble of electrons to change its quantum state at low temperatures, and for the existence of permanent currents in a superconducting wire.

## I. INTRODUCTION

The basic phenomenon of superconductivity in metals is the formation of weakly bound states between pairs of electrons. These states are often considered as a surprising phenomenon in view of the fact that two particles of equal charge should repel each other. Where does the attraction between electrons come from? This attraction has long since been explained by Bardeen, Cooper and Schrieffer as an effect of the interaction between the electrons and the lattice. However, these calculations do not appeal to direct physical intuition. In this essay we propose a simple physical picture of the effects that bring about that attraction which is easily visualized. Our description is fundamentally identical with the BCS theory; it is formulated in a way which may make it easier to see how an attraction between oppositely moving electrons comes about. It is shown that our formulation leads to the same expression of the binding energy of Cooper pairs as the usual calculations.

We also discuss in simple terms how the ensemble of Cooper pairs forms a Bose gas, whose lowest quantum state is stable as long as the energy transfer per electron pair is smaller than the binding energy  $\Delta$  of the Cooper pair. This state, therefore, will be dominant as long as  $kT < \Delta$ . Furthermore, it will be shown why this state leads to permanent currents in the presence of magnetic fields. Indeed, the magnetic field produced by the current in a superconducting wire is just sufficient to maintain the current that produced it.

The presentation does not contain any new ideas as to the theory of superconductivity. It attempts to formulate the theory in such a way that the physical effects become more apparent. Only Chapters III to VI contain new pedagogical ways of presenting the situation. Chapter VII, regarding the properties of a current in a superconducting wire, does not make use of any unconventional formulations.

## II. SEMIQUANTITATIVE RELATIONS

We consider a metal as a cubic lattice of positive ions (charge +1) filled with a degenerate gas of free electrons at zero temperature. The lattice distance  $d$  is of the order of a few Bohr radii. Therefore we observe the approximate identity<sup>\*</sup>):

$$\epsilon_A \equiv \frac{e^2}{d} \sim \frac{h^2}{md^2}, \left[ \frac{e^2}{d} = f \frac{h^2}{md^2}, f > 1 \right] \quad (1)$$

$$f = \frac{d}{a_B}. \quad (1a)$$

Here  $m$  is the mass of the electron, and  $a_B$  is the Bohr radius<sup>\*\*</sup>). The constant  $f$  is somewhat larger than unity.  $\epsilon_A$  is an atomic energy of the order of a few electron volts. The momentum of the fastest electrons on top of the Fermi distribution is

$$p_F \sim \frac{h}{d} \left[ p_F = a \cdot \frac{h}{d}, a = 3.1 \right] \quad (2)$$

and therefore their velocity is

$$v_e \sim \frac{h}{md} \left[ v_e = a \cdot \frac{h}{md} \right]. \quad (2a)$$

This velocity is of the order of the speed of electrons in atoms.

We are going to describe the mechanics of the lattice in the simplest possible terms: every ion is an independent oscillator with mass  $M$  and frequency  $\omega_D$  (the Debye frequency). Let us determine this frequency. We expect the potential energy  $\frac{1}{2}M\omega_D^2x^2$  of a displacement  $x$  to be  $\sim\epsilon_A$  if  $x \sim d$ . Hence we get from (1):

\*) Here and in the following we deal with approximate equalities. In order to follow up the accumulation of approximate relations, we introduce numerical constants of the order unity such as  $f$ . At the end we will be able to estimate the combinations of such quantities as they appear in the final results. The "exact" expressions containing these constants are put in square brackets.

\*\*) Throughout this paper the letter  $h$  stands for  $\hbar$ , that is Planck's constant divided by  $2\pi$ .

$$\frac{1}{2}M\omega_D^2 d^2 \sim \epsilon_A \sim \frac{h^2}{md^2}$$

$$\omega_D \sim \sqrt{\frac{m}{M}} (\epsilon_A/h) \quad (3)$$

$$\left[ \omega_D = b \frac{h}{\sqrt{mMd^2}} \right] \quad (3a)$$

where b is a number somewhat larger than unity<sup>\*)</sup>. The square root of the mass ratio will play an important role; we call it

$$\beta \equiv \sqrt{\frac{M}{m}} \sim 300 . \quad (3b)$$

By the way,  $\beta$  is the "Mach number" of the electron motion, since the sound velocity obviously is  $v_s \sim d \cdot \omega_D$ .

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\*) In deriving the magnitude of  $\omega_D$ , Eq. (3) really should read  $\frac{1}{2}M\omega_D^2 (d/2)^2 \sim \epsilon_A$ , since a displacement by  $d/2$  should already give rise to energy charges of the order  $\epsilon_A$ . This would make  $b \sim 8$ . Indeed  $b$  lies between 5 and 10 in most metals.

### III. THE POTENTIAL TUBE BEHIND THE ELECTRON

We now study the effect of the electron motion on the lattice. Consider an electron on top of the Fermi distribution. It spends the time

$\tau = \frac{d}{v_e}$  within a distance  $\sim d$  from a given ion. During this time it transfers a momentum

$$p \sim \tau e^2/d^2 \quad (4)$$

$$[p = c \cdot \tau e^2/d^2]$$

to a neighboring ion, in the direction towards the electron ( $c$  is a constant near unity). Thus, the ion moves towards the electron path by a distance  $\delta$ , calculable from the dynamics of an oscillator<sup>\*)</sup>:  $\delta = p/(M\omega_D)$ . We

therefore get from Eqs. (2), (3) and (4)

$$\delta \sim \frac{d}{v_e} \frac{e^2}{d^2} \frac{1}{M\omega_D} \sim \frac{1}{v_e} \frac{e^2}{d} \frac{h}{\sqrt{mM} \epsilon_A} \sim \sqrt{\frac{m}{M}} d \sim \frac{d}{\beta} \quad (5)$$

$$\left[ \delta \approx \frac{fc}{ab} \frac{d}{\beta} \right].$$

Thus the ions close to the electron path are displaced by  $\delta$  toward the path. They will not stay forever in this displaced position; they will again return to their position after a time of the order  $\omega_D^{-1}$ , say  $s \cdot \omega_D^{-1}$ . Again  $s$  is a numerical constant of order unity. Therefore the displacements of the ions

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\*) This calculation is valid only for a classical oscillator. Such a classical calculation may be questionable, since the amplitude of the zero-point oscillation  $\delta_0 \sim \beta^{-1/2} d$  is larger than the displacement  $\delta$ .

However, as frequently happens with oscillators, a quantum mechanical calculation of the effect of a momentum transfer  $p$  to an oscillator in its ground state, gives the same displacement  $\delta$  as the classical one.



towards the path extend over a distance  $\ell$  behind the electron\*) which is given by (see (2) and (3))

$$\ell \sim \frac{v_e}{\omega_D} \sim \beta d \quad (6)$$

$$\left[ \ell = s \frac{v_e}{\omega_D} = s \frac{a}{b} \beta d \right]$$

The displacements of the positive ions towards the path of the electron create a negative (attractive) potential  $U$  for any other electron. This potential is contained within a thin tube of length  $\ell$  and of diameter  $\sim d$ , and follows the electron when moving through the lattice (see Fig. 1).

Let us now determine the potential  $U$  in that tube relative to the average potential within the crystal. A displacement  $\delta$  changes the ordinary potential  $\sim -e^2/d$  of an ion at a distance  $\sim d$  by

$$U \sim - \frac{e^2}{d^2} \delta \sim \frac{e^2}{d} \frac{1}{\beta} \quad (7)$$

$$\left[ U = - \kappa \frac{1}{\beta} \frac{e^2}{d}, \kappa = \frac{fcq}{ab} \sim 1 \right]$$

This follows from (5);  $g$  is a new numerical constant larger than unity since the average distances from the ions are less than  $d$ , and there is more than one ion near a given point in the tube.  $f$ ,  $b$  and  $a$  are also larger than unity, so that we expect  $\kappa \sim 1$ . Thus the electron produces behind itself a potential tube of length  $\ell$  and diameter  $d$  with a small negative potential (7). Note that, according to (7), (3b), (3) and (1):  $-U \sim \hbar \omega_D$ .

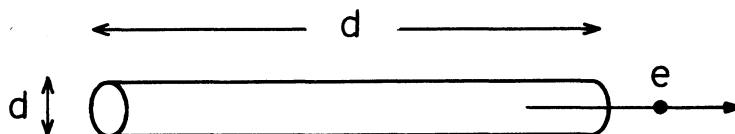


Fig. 1 There will be a negative potential behind the electron  $e$  moving towards the right, within a cylinder of length  $\ell$  and diameter  $\sim d$ .

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\*) In order to understand the argument we must be aware that  $T=0$  and, therefore, the electrons cannot transfer energy to the ion vibrations. The ions return to their original state after a time of the order  $\omega_D^{-1}$  from the encounter with the electron. The displacements  $\delta$  do not spread over the lattice as sound waves. They are part of the quantum state of the electron in a lattice which is different from that of a free electron. The electron is accompanied by a "tail" of ions displaced towards its path. This is a property of the "quasi-particle" which the electron represents when it moves in a lattice.

This tail contributes to the effective mass of the electron. A simple estimate of the energy contained in the tail of displaced ions shows that it is of the order of  $\beta^{-1} \epsilon_A$ ; that is negligible compared to the electron mass.

#### IV. THE COOPER PAIRS

How does this potential tube contribute to the interaction of two electrons? Only the electrons near the Fermi surface can make use of such weak interactions since the ones below are "frozen"; they cannot change their quantum state. In order to feel the full extent of the potential, they must move head-on coming from opposite directions. By "head-on" we understand a motion in which the closest distance of approach is equal to or less than  $d$ . But  $d$  is of the order of the wavelength  $\lambda$  of the electrons on the Fermi surface. Thus the two electrons must be in a relative S-state ( $L=0$ , where  $L$  is the quantum number of their relative orbital angular momentum.) If they had a relative  $L \neq 0$ , their closest distance of approach would be  $\sim L\lambda$ . Since  $\lambda \sim d$  they would miss their mutual potential tubes if  $L \neq 0$ . Thus only electron pairs in S-states are subject to this potential to its full extent<sup>\*)</sup>.

We therefore express the mutual potential in the form (see Fig. 2)

$$V(r) \sim \begin{cases} = -\frac{1}{\beta} \kappa \frac{e^2}{d} \delta_{L0} & \text{for } r < \ell \\ \sim 0 & \text{for } r > \ell \end{cases} \quad (8)$$

where the delta-function indicates that the potential acts only in S-states<sup>\*\*)</sup>. We now understand the physical reason why the potential of the lattice deformation binds electron pairs and why it does so only in relative S-states. A further consequence is the fact that the two electrons must have opposite spin, in order to fulfil the Pauli principle, since the S-state is symmetric.

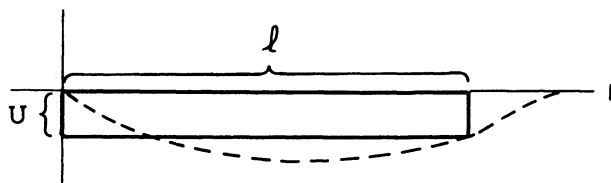


Fig. 2 The potential within the tube of Fig. 1, as function of the distance  $r$  behind the moving electron. This is the potential felt by another electron. The solid line is our approximation; actually it will have a form similar to the broken line, since the ions take some time after the passing of the electron before they move the full distance  $\delta$ .

\*) It may seem that two electrons moving in almost the same direction should also be subject to this potential. Appendix I shows why this is not so.

\*\*\*) Here we face a peculiar situation: The potential acts only when the electrons leave each other (that is, when  $\dot{r} > 0$ ) and not when they approach. This is because the "tails" trail the electrons. Hence we do not use a factor 2 in (8) coming from the effects of both electrons.

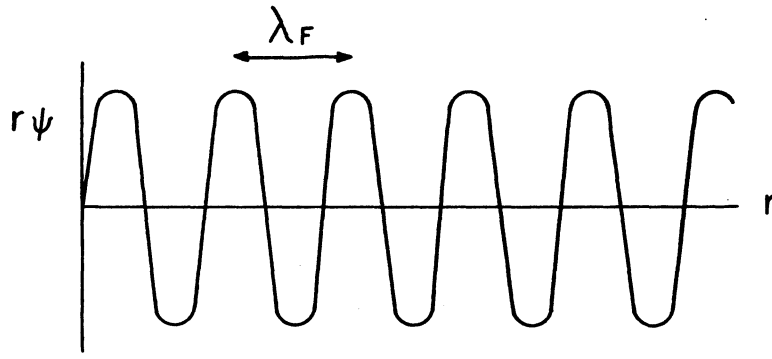


Fig. 3a Shows the function  $r\psi$  of two non-interacting electrons near the Fermi surface in a relative S-state as function of their distance  $r$  without any interaction.

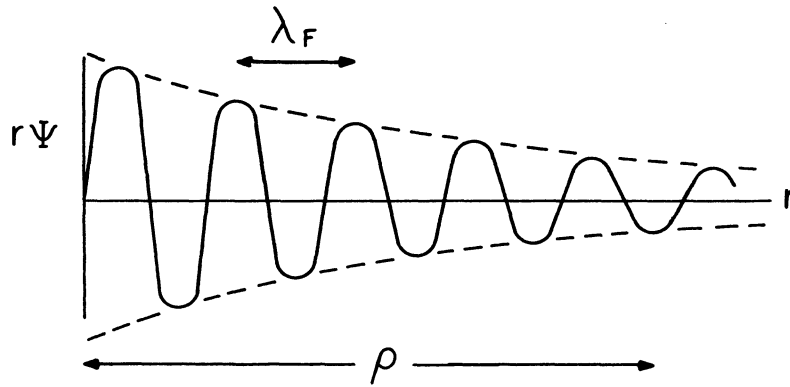


Fig. 3b Shows the wave function  $r\Psi$  of two electrons with interaction. This is the wave function of a Cooper pair.  $\lambda_F$  is about twice the lattice distance,  $\rho$  is the coherence length. Note that, actually,  $\rho$  is about  $10^4$  lattice distances.

We now show that this attraction indeed gives rise to bound pairs. The motion of a bound pair in an S-state can be visualized as a head-on back and forth movement; the direction of this movement is distributed uniformly with equal probability over all possible directions. Positronium in its ground-state and the deuteron are examples (see Fig. 3). The hydrogen atom in S-states is also an example; in that case it is the electron that does most of the back and forth moving, whereas the proton only recoils slightly. In order to determine the bound state of the Cooper pair, we must solve the Schrödinger equation for the potential (8). We know that the solution  $\psi(r)$  must be an S-state ( $r$  is the relative distance of the two electrons); therefore we expand  $\psi$  in terms of S-states  $\psi(p)$  of a pair of free electrons with a relative momentum  $p$  and a total momentum  $P=0$ , since they move against each other. (Our wave functions are  $r$  times the three-dimensional ones. This reduces the solution to a one-dimensional problem with the boundary condition  $\psi(0) = 0$ ):

$$\psi(p) = \sqrt{\frac{2}{R}} \sin \frac{pr}{h} . \quad (9)$$

These functions are normalized and orthogonal in a large sphere of radius  $R$ . We then write for the wave function  $\Psi(r)$  of the Cooper pair:

$$\Psi(r) = \sum_{p'} \alpha(p') \sqrt{\frac{2}{R}} \sin \frac{p'r}{h}, \quad (10)$$

where the sum extends over all possible values  $p'$  of  $p$  in the sphere  $R$ . The Schrödinger equation for  $\Psi$  is

$$\left( \frac{1}{2\mu} p^2 + V \right) \Psi = E \Psi, \quad (11)$$

where  $\mu = m/2$  is the reduced mass.

We are interested in the binding effect of the potential (8) upon a pair on or very near to the Fermi surface. The binding will reduce their energy  $2\varepsilon_F$  to  $E = 2\varepsilon_F - \Delta$ , where  $\varepsilon_F$  is the Fermi energy. Thus,  $\Delta$  is the binding energy of the Cooper pair. It turns out, as it will be shown below, that  $\Delta$  is given by

$$\Delta \sim h\omega_D \exp \left( -\xi 2\pi \frac{\varepsilon_F}{e^2/d} \right) \sim 10^{-4} \varepsilon_A,$$

where  $\xi$  is a numerical constant. This, indeed, is the expression which is usually obtained by a more complicated calculation of emissions and absorptions of sound waves. In solving the Schrödinger equation, care must be taken that the sum in Eq. (10) does not contain any momenta  $p < p_F$ , since they are occupied by other electrons. The calculation will show that the interval  $\Delta p$  over which the coefficients  $\alpha(p)$  are decisively different from zero, is rather small:

$$\Delta p \sim \frac{m}{p_F} \Delta.$$

We now proceed to the actual derivation of the expression for the binding energy from the Schrödinger equation (11). Those who are not interested in this calculation can skip it until after Eq. (23). We insert Eq. (10) into Eq. (11), multiply with  $\psi(p)$ , and integrate; the result is

$$\alpha(p) \left( \frac{p^2}{m} - E \right) = - \sum_{p'} \alpha(p') (p|V|p'), \quad (12)$$

where  $(p|V|p')$  is the matrix element of the potential  $V$ :

$$(p|V|p') = \int_0^R dr \psi(p) V(r) \psi(p'), \quad (13)$$

with  $V(r)$  given by Eq. (8), and  $E$  is the eigenvalue of the energy in the bound state  $\Psi$ .

Let us now introduce an energy variable  $\epsilon$  instead of  $p$ :

$$\epsilon = \frac{1}{m} \left( p^2 - p_F^2 \right) . \quad (14)$$

The zero-point of this energy scale is the Fermi energy  $2\epsilon_F$  of the pair ( $\epsilon_F = p_F^2/2m$ ); that is the energy the pair would have without the interaction  $V(r)$ . In this energy scale the eigenvalue  $E$  will be the negative of the binding energy  $\Delta$  of the pair,

$$E = -\Delta . \quad (15)$$

Then we get from Eq. (12) and Eq. (15)

$$\alpha(\epsilon)(\epsilon+\Delta) = \sum_{\epsilon'} \alpha(\epsilon')(\epsilon|V|\epsilon') . \quad (16)$$

Let us now calculate the matrix element Eq. (13). Since  $V(r) = 0$  for  $r > \ell$ , and a constant for  $r < \ell$ , the integral reduces to an integration from 0 to  $\ell$  of the product

$$\psi(p)\psi(p') = \frac{1}{2} |\cos(p+p')r/h - \cos(p-p')r/h| .$$

The values of  $p$  are always close to  $p_F$ ; hence the integration of the periodic functions over a length  $\ell \gg \frac{h}{p_F}$  will vanish except when  $p-p'$  is of the order  $(h/\ell)$  or smaller. We then simplify the result to

$$\int_0^\ell \psi(p)\psi(p')dr \begin{cases} = -\frac{\ell}{R} & \text{for } p-p' < \frac{h}{\ell} \\ = 0 & \text{for } p-p' > \frac{h}{\ell} \end{cases} . \quad (17)$$

What interval  $\Delta\epsilon$  of  $\epsilon$  corresponds to  $p-p' = h/\ell$  near  $p = p_F^2$ ? We see from

Eq. (14) that  $\epsilon - \epsilon' = \frac{2p_F(p-p')}{m}$ , so we get for the critical energy difference  $\Delta\epsilon$ ,

$$\Delta\epsilon = \frac{2p_F}{m} \frac{h}{\ell} \sim \frac{1}{\beta} \frac{2p_F}{m} \frac{h}{d} \sim \frac{1}{\beta} \frac{2h^2}{md^2} \sim h\omega_D \quad (18)$$

$$\left[ \Delta\epsilon = \eta h\omega_D , \quad \eta = \frac{2a}{\beta b} \approx 1 \right] .$$

Since  $\eta$  seems to be very close to unity, we will omit it in what follows.

We determine the matrix element from Eqs. (6), (8), (17) and (18):

$$(\epsilon|V|\epsilon') \sim \begin{cases} \frac{1}{\beta} \frac{e^2}{d} \frac{\ell}{R} \sim \frac{e^2}{R} & \text{for } \epsilon - \epsilon' < \Delta\epsilon \sim \omega_D \\ 0 & \text{for } \epsilon - \epsilon' > \omega_D \end{cases} \quad (19)$$

$$\left[ (\epsilon|V|\epsilon') = \kappa \frac{sa}{b} \frac{e^2}{R} \quad \text{for } \epsilon - \epsilon' < \omega_D \right] .$$

It is interesting that the factor  $\beta$  disappears from the matrix element.

Now we come to an important point. The Schrödinger equation (16) would be correct if the two electrons were isolated. But they sit on the top of a Fermi sea. Therefore no states with an energy below the Fermi surface are available for the formation of the Cooper pair. Hence the sum over  $\epsilon'$  is restricted to values  $\epsilon' > 0$ ! Let us transform the sum  $\sum_{\epsilon'}$  into an integral.

The number of S-states in an energy interval  $d\epsilon$  near  $\epsilon_F$  is  $\frac{mR}{2\pi p_F \hbar} d\epsilon$  and therefore the restricted Schrödinger equation (16) becomes <sup>\*)</sup> (we have put  $\eta = 1$  in Eq. (16))

$$\alpha(\epsilon)(\epsilon + \Delta) = -B \int_0^{h\omega_D} \alpha(\epsilon') d\epsilon' \quad (20)$$

$$B \sim \left. \begin{aligned} \frac{e^2 m}{2\pi p_F \hbar} &= \frac{e^2 m p_F}{2\pi p_F^2 \hbar} \sim \frac{1}{2\pi} \frac{e^2}{d} / \epsilon_F \\ \left[ B = k \frac{e^2 m}{2\pi p_F \hbar} = \frac{ska^2}{2b} \frac{1}{2\pi} \frac{e^2}{d\epsilon_F} \right] \end{aligned} \right\} \quad (21)$$

In the determination of B the relations  $\epsilon_F = p_F^2/(2m)$  and  $p_F = a(h/d)$  were used. We have retained the numerical factor  $2\pi$  in the approximative expression of B, since it will appear in an exponent at the end. The right-hand side of Eq. (20) is independent of  $\epsilon$ ; therefore we set

$$a(\epsilon) = (\epsilon + \Delta)^{-1} \quad (22)$$

and get

$$1 = -B \int_0^{h\omega_D} \frac{d\epsilon'}{\epsilon' + \Delta} = -B \log \frac{\eta h\omega_D + \Delta}{\Delta}.$$

It will turn out that  $\Delta \ll h\omega_D$  so that we get

$$\Delta = h\omega_D \exp \left( -\xi \frac{2\pi\epsilon_F}{e^2/d} \right) \quad (23)$$

$$\xi = \frac{2b}{ska^2} = \frac{2b^2}{safcg}.$$

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\*) The upper limit of the integral should have been  $\sim(h\omega_D + \epsilon)$  since the matrix elements are different from zero for  $\epsilon - \epsilon' \sim h\omega_D$ . It will turn out, however, that  $\alpha(\epsilon')$  is significantly different from zero only for  $\epsilon \ll h\omega_D$ , so that we may forget the dependence on  $\epsilon$  of the upper limit.

The factor  $\xi$  is of the order unity;  $b^2$  cancels the effects of  $f$  and  $g$ .  $\Delta$  is the binding energy of the Cooper pair. Let us call

$$\gamma = \exp\left(-\xi 2\pi \frac{\epsilon_F}{e^2/d}\right). \quad (24)$$

It is of the order of  $10^{-2}$  to  $10^{-3}$  since  $\epsilon_F/(e^2/d) \sim 1$ . We then can express the ratio of  $\Delta$  to atomic energies as follows (see Eq. (3)):

$$\Delta = \beta\gamma\epsilon_A. \quad (25)$$

This is about  $10^{-4}$  to  $10^{-5}$  smaller than atomic energies. Thus we arrive at roughly a few  $10^{-4}$  eV, which corresponds to  $kT$  of a few degrees.

It is interesting to observe that two isolated electrons with energies corresponding to the top of the Fermi distribution would never be bound by the potential well Eq. (8). It is easily seen that for a pair of isolated electrons, this potential gives rise to a lowest bound state with a binding energy  $\Delta \sim |V(0)|$ . The kinetic energy in this state would be small (factor  $\beta^{-1}$  compared to the potential energy  $V(0)$ ), because of the fact that  $\ell$  is so large. However  $V(0) \sim \frac{1}{\beta} \epsilon_F$ , so that, at Fermi energies there would be no bound states; it would be high in the continuum. It is the restriction to states with an energy  $E$  larger than the Fermi energy that leads to binding.

There is another interesting fact that helps to understand the particular form Eq. (23) of the binding energy. It is well known that a potential well in three dimensions (depth  $U$ , radius  $\ell$ ) does not give rise to binding until  $U$  is larger than

$$U > t \frac{\hbar^2}{m\ell^2},$$

where  $t$  is some numerical constant. A one-dimensional well, however, binds for any value of  $U$ . The binding energy of the lowest state is

$$\Delta = t' \frac{U^2}{\hbar^2/(m\ell^2)},$$

with  $t'$  another numerical constant. A well in two dimensions also gives rise to a bound state at any value of  $U$ :

$$\Delta = t'' \frac{\hbar^2}{m\ell^2} \exp\left[-\xi \frac{\hbar^2/(m\ell^2)}{U}\right],$$

where  $t''$  and  $\xi$  are numerical constants. Here the binding goes to zero exponentially with  $U \rightarrow 0$ . This form of  $\Delta$  reminds us of the expression Eq. (23) for the Cooper pair. Indeed, the two cases are not dissimilar since the case of the Cooper pair is a problem restricted to two dimensions (the Fermi surface) in the momentum space.

V. THE WAVE FUNCTION OF A COOPER PAIR

Since the wave function of a Cooper pair represents a bound S-state, the motion it describes is a periodic back and forth movement of the two electrons in directions which are uniformly distributed, covering a relative distance  $\rho$  between them as sketched in Fig. 3. It is analogous to the motion of the two nucleons in a deuteron or of the two electrons in the ground state of positronium. The wave function is finite; it goes to zero much faster than  $r^{-1}$  at distances  $r > \rho$ .

Let us now determine  $\rho$  from the momentum spread in the wave function. According to Eq. (9a) and Eq. (22) the wave function of the Cooper pair consists of waves of the type (9) (see Fig. 4) within a very narrow interval of  $p$  close to  $p_F$ . Quite generally, the average spread of momentum  $\Delta p$  of a bound state is connected with its binding energy  $\Delta$  by the relation

$$\Delta = \frac{p_F}{m} \Delta p . \quad (26)$$

Since  $\epsilon_F = p_F^2/2m$ , we get  $\Delta p/p_F \sim \Delta/\epsilon_F$ , an extremely small quantity. Thus  $\Psi$  consists mainly of waves of wave number  $p_F/h$ . It will look like  $\sin(p_F r/h)$  for  $r \ll \rho$ , where  $\rho$  is the distance at which the different  $p$ 's begin to interfere. For  $r > \rho$  the waves (9) will destroy themselves by interference.

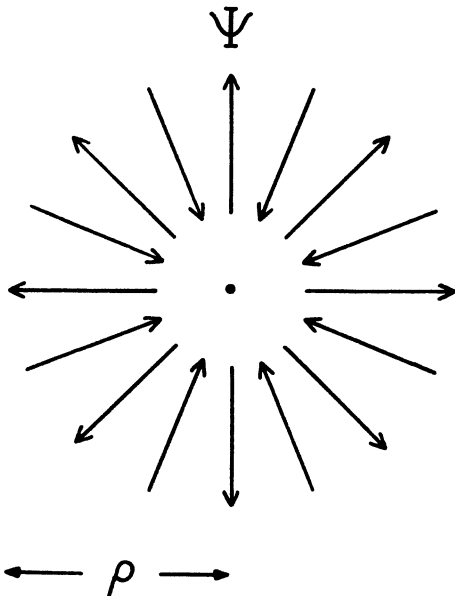


Fig. 4 A picture of the dynamics in the quantum state of the Cooper pair. It is a linear combination of motions away and towards one another. The electrons stay within a distance of order  $\rho$ .



Thus  $\Psi$  will have a finite extension (see Fig. 4). The distance  $\rho$  is the "size" of the wave function  $\Psi$ . Clearly,  $\rho = h/\Delta p$ , and we get

$$\rho \sim \frac{hp_F}{m\Delta} \sim \frac{p_F^2}{m\Delta} d \sim \frac{\epsilon_F}{\Delta} d, \quad (27)$$

where the second " $\sim$ " sign comes from  $p_F \sim h/d$ , and the third from  $(p_F^2/m) \sim \epsilon_F$ . Here we see that  $\rho$  is very large compared to the lattice distance. According to Eq. (25), it is even larger by a factor  $\gamma^{-1}$  (see Eq. (2)) than the length  $\lambda \sim \beta d$  of the potential tube. A wave function that spreads over distances much larger than the binding potential is a well-known phenomenon when the binding energy is small; remember the deuteron.

The large extension of the wave function also explains why the electrostatic repulsion between the electron pair does not appreciably influence the binding. That repulsion acts only over distances of the order  $d$  since the electric field at larger distances (outside the "Debye length") is strongly reduced by the rearrangement of the electron gas. The remaining repulsion will decrease the wave function within a distance of a few  $d$  from the center. Since  $\rho \gg d$ , that effect changes the wave function only very near the center and does not influence the binding energy to any appreciable extent since the potential  $V(r)$  reaches to much larger distances,  $\lambda \gg d$ . The length  $\rho$  is called the "coherence length" in the theory of superconductivity. It is of the order of  $10^{-4}$  cm.

VI. THE ENSEMBLE OF COOPER PAIRS

Cooper pairs are formed only among those electrons that lie in the uppermost part of the Fermi distribution, not lower than  $\Delta$  below the surface  $\epsilon_F$  in energy, or  $\Delta p$  as defined by Eq. (26) below the surface  $p_F$  in momentum, (see Fig. 5). Electrons that lie lower than that cannot make use of empty states in order to form the Cooper-pair wave function  $\Psi$ . All Cooper pairs possess the total momentum  $P=0$ . They are, therefore, in the same quantum state. This does not violate any statistics since, as two-electron systems, they obey Bose statistics; they form a degenerate Bose gas.

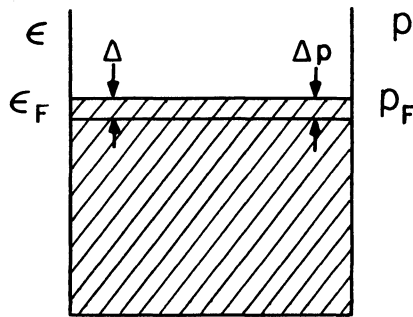


Fig. 5 The electron distribution in a metal. All states below  $\epsilon_F$  (energy scale) or  $p_F$  (momentum scale) are occupied. The electrons in the interval  $\Delta$  (energy) or  $\Delta p$  (momentum) from the upper limit are forming Cooper pairs.

What is the number  $n'$  of Cooper pairs per  $\text{cm}^3$ ? Call  $n = \frac{1}{d^3}$  the number of electrons per  $\text{cm}^3$ ; then we get

$$n' \sim \frac{1}{2} n \frac{\Delta}{\epsilon_F}$$

since our relations are only semiquantitative, and we can omit the factor  $\frac{1}{2}$ . We then may write

$$n' \sim \frac{1}{d^3} \frac{\Delta}{\epsilon_F} \sim \frac{1}{d^2 \rho}$$

and get for the distance  $D$  between Cooper pairs:

$$D \sim \left( \frac{1}{n'} \right)^{1/3} = d \left( \frac{\rho}{\Delta} \right)^{1/3} \sim (30-40)d . \quad (28)$$

Since the size  $\rho$  of a pair is much bigger than that, they overlap appreciably. It is important to observe, however, that they still remain orthogonal, in spite of their overlap:

$$\int \psi_i \psi_k dt = \delta_{ik}, \quad (29)$$

where  $dt$  is the coordinate space of the electrons included in the pairs. This is not so surprising since the wave functions oscillate around the zero value as indicated in Fig. 3. We can see it by the following semiquantitative consideration. When the interaction between the electrons is "switched off" the electrons involved in the Cooper pairs are in orthogonal states and fill the phase space in the upper part of the Fermi distribution to about  $\Delta$  below the Fermi surface. The "switching on" of the interaction does not change their energy or momentum appreciably; indeed only by amounts of the order  $\Delta$  in energy or  $\Delta p$  in momentum. Thus the phase space of those electrons is not changed very much; since they have been in orthogonal states and have filled the phase space fully and completely before switching on the interaction, they will do so afterwards too to a good approximation. That means they will remain orthogonal (Eq. (29)) and, moreover, there will be almost no place in the phase space for additional Cooper pairs without violating condition (29). The Cooper-pair wave functions are another set of orthogonal states for those  $n'$  electrons in the energy interval  $\Delta \epsilon$  below the Fermi surface.

In order to understand the nature of this ensemble of Cooper pairs, let us consider for a moment a dilute and a "compact" degenerate Bose gas of He atoms. We understand by dilute, a gas with a density (number of atoms per  $\text{cm}^3$ )  $n \ll \frac{1}{r_0^3}$ , where  $r_0$  is the radius of the He atom. At zero temperature

all atoms will be in lowest state of motion in a container  $\Omega$ . Fig. 6 shows the states of translatory motion of an atom in  $\Omega$ ; they are infinitesimally near to each other. The lowest state is occupied by all He atoms. But one

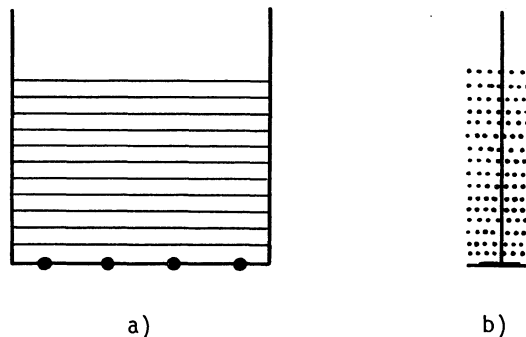


Fig. 6 Dilute Bose gas. (a) The single-particle states. At zero temperature all particles are in the ground state (no motion). (b) The states of the ensemble. At zero temperature the gas is in the lowest state, but there are states available in the immediate vicinity above the lowest state, corresponding to one or a few particles being in those states of (a) that correspond to a slow motion.

atom could easily go to the next higher state. Although it then would move relative to the others, it would have very little chance to collide with the other atoms. Its state would essentially be a state of free motion. It would need an infinitesimal energy for one atom to break loose and go into a state of different speed. We call that a situation with no gap.

We now consider a "compact" gas which we understand to be a gas in which the atoms touch one another (see Fig. 7). They still do not overlap, the condition (29) is valid for them. But the addition of a few more would lead to a violation of Eq. (29). The degenerate state of this Bose gas is a state in which all atoms have the same momentum  $P$ , for example  $P=0$ . In this situation it costs energy to "break loose" for one atom and acquire a different

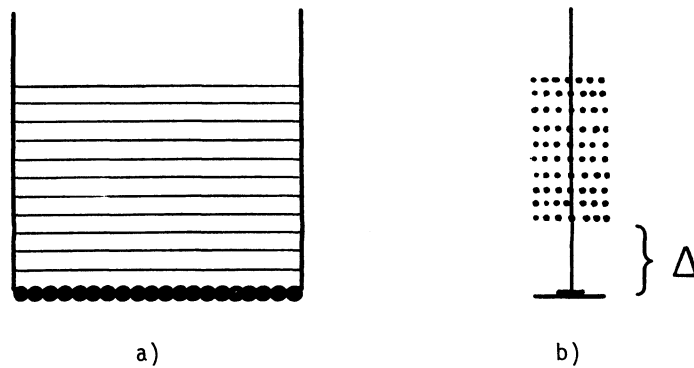


Fig. 7 Compact Bose gas. (a) The single-particle states. At zero temperature all particles are in the ground state; they touch each other. (b) The states of the ensemble. At zero temperature the gas is in the lowest state. The next higher state is at a finite distance  $\Delta$  above the lowest state, because it is not possible to excite one or a few particles to a motion different from the others. The only way to excite the ensemble is to produce rotons, or to put a particle into an excited state.

velocity from the rest. We can understand the situation by comparing it with a completely filled subway car. There are two ways for a person in the middle of the crowd to move relative to the others in order to get to the exit: the first way is to penetrate through your neighbors. Of course, this is impossible for a passenger, but a He atom can do it by going over to an excited state of the atom; then the Pauli principle no longer forbids such an atom to occupy the same space with one in the ground state. It is an expensive proposition, since it would cost an atomic energy  $\epsilon_A \sim h^2/(mr_0^2)$  to do so. The other much cheaper (and also more realistic) way is to change places with a neighbor by executing a common rotation by  $180^\circ$  with him or with a small group of neighbors (Fig. 8). This type of motion symbolizes the formation of a roton. According to quantum mechanics the minimum energy of a rotational motion of two bodies with a mass  $M$  and a distance  $r_0$  is of the order of  $h^2/(Mr_0^2)$  which, for helium atoms, is  $(m/M)$  times smaller than atomic energies; it is of the order of a few  $10^{-4}$  eV, which corresponds to a temperature of a few degrees. Thus, the breaking loose of an atom from the compact Bose gas

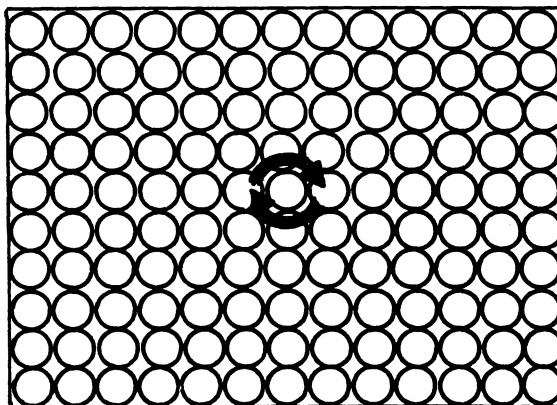


Fig. 8 A view of a compact Bose gas. A particle can be displaced only by a rotation of a group or by excitation.

(all particles in the same state) costs a finite amount of energy. We have a finite energy gap between the degenerate Bose-gas state where all atoms have the same momentum, and a state where one atom is broken loose. The energy gap for the formation of rotons is the transition temperature of a superfluid. If  $kT$  is less than this gap, the fluid can move only as a whole.

Let us return to the gas of Cooper pairs. It also is a compact gas, since further additions of pairs would violate Eq. (29). When all pairs are in the same quantum state  $P=0$ , it would need a finite energy to break loose a pair and to give it a different momentum. In the case of the Cooper pairs, however, the excitation energy  $\Delta$  (the energy to dissociate a pair) is much smaller than a roton energy, since the total mass of the pair is only  $2m$ . (Indeed, the roton energy would be  $\sim \epsilon_A$ .) Thus, here it is  $\Delta$  that determines the gap. Indeed  $\Delta$  corresponds to  $kT$ , with  $T$  being of the order of the transition temperatures of superconductors. As long as  $kT \ll \Delta$ , the Cooper pairs form a degenerate Bose gas.

The formation of this compact degenerate Bose gas acts like a "crust" on top of the Fermi distribution, in the sense that it takes an energy  $\Delta$  to liberate an electron from the Cooper-pair gas, and even more energy to liberate it from the electron distribution below the strip in Fig. 5, because you would have to lift it to empty states above the strip. This strip contains the "frozen" ensemble of Cooper pairs. Thus all electrons, including those that do not make up the Cooper pairs, form a collective state with total momentum  $P=0$ , a state that does not break up as long as  $kT < \Delta$ .

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\*) The free electrons below the strip are distributed in such a way that for any electron with a momentum  $\vec{p}$ , there is one with the opposite momentum  $-\vec{p}$ . Thus, their total momentum is zero.

## VII. THE SUPERCURRENT

How does it happen that such an ensemble of electrons with total momentum  $P=0$  ever produces a current and a supercurrent without any resistance at all? The reason is that, in the presence of a magnetic field, the velocity is no longer proportional to the momentum. The general definition of velocity is

$$\vec{v} = \frac{\partial k}{\partial \vec{p}},$$

where  $k$  is the kinetic energy which, for free electrons, in the presence of a vector potential  $A$ , is given by

$$k = \frac{1}{2m} \left( \vec{p} - \frac{e\vec{A}}{c} \right)^2. \quad (30)$$

Hence we get

$$\vec{v} = \frac{1}{m} \left( \vec{p} - \frac{e\vec{A}}{c} \right) \quad (31)$$

for the velocity. The current density  $j$  produced by our ensemble of electrons is then

$$\frac{1}{c} \vec{j} = \frac{e}{c} \sum_i \vec{v}_i = \frac{e}{mc} \sum_i \vec{p}_i - \frac{e^2 n}{mc^2} \vec{A}, \quad (32)$$

where  $n$  is the number of electrons per unit volume. Here the slight changes of momentum caused by the bindings of the Cooper pairs have been neglected; anyway, most of the current comes from the electrons below the Cooper-pair crust. The sum over the momenta in Eq. (32) vanishes since  $P=0$ , and we get

$$\frac{1}{c} \vec{j} = - \frac{1}{\lambda^2} \vec{A}, \quad \lambda = \left( \frac{mc^2}{e^2 n} \right)^{1/2} = d \left( \frac{d}{r_0} \right)^{1/2}, \quad r_0 = \frac{e^2}{mc^2}. \quad (33)$$

Here we see direct proportionality<sup>\*)</sup> between current and  $A$ ; the constant is the reciprocal square of a length  $\lambda$  which contains  $d$  from  $n=d^{-3}$  and the so-called classical electron radius  $r_0$ . It is easily seen that

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\*) One may wonder how it is possible that a physical magnitude such as the current density is proportional to a non-gauge-invariant magnitude  $\vec{A}$ . The answer is this. A change in gauge would transform  $\vec{A}$  into  $\vec{A}' = \vec{A} + \eta$ , where  $\eta$  is a function of space. This changes the wave function of the electrons from  $\psi$  into  $\psi' = \psi \exp \left[ \frac{ie}{c} \eta \right]$ ; therefore it also changes the momenta  $\vec{p}$  of all electrons to  $\vec{p}' = \vec{p} + (e/c)\eta$ , thus cancelling the change of  $A$  in Eq. (30). We choose that natural gauge in which the sum of all momenta of the electrons remains zero.

$$\lambda = f^{\frac{1}{2}} \cdot \frac{hc}{e^2} d ,$$

where  $f$  is as defined in (1a). It is a little more than 137 times the lattice distance\*).

It seems paradoxical that an electric current appears in a system in which the total momentum remains zero. The following example may illustrate the situation. Let us look at the electron shell of a neon atom. It has a filled shell; therefore the total angular momentum is zero, although some electrons circulate one way and an equal number the other way, with one unit  $h$  of angular momentum\*\*). When a magnetic field  $\vec{B}$  is switched on ( $\vec{A} \neq 0$ ), the electrons whose angular momentum  $\vec{L}$  is parallel to  $\vec{B}$  circulate faster, and those with opposite  $\vec{L}$  circulate slower (Zeeman effect), but the values of their angular momentum remain the same (after all, it can only be an integer multiple of  $h$ ). Thus, we get a circular current without a change of the angular momentum whose total value remains zero. The circular current produces an additional magnetic field and this phenomenon is called diamagnetism.

It must be realized how important is the existence of the energy gap  $\Delta$  for the understanding of the supercurrent. The current (33) is nothing else than the induction current which appears when a magnetic field is switched on. The process of switching on accelerates the electrons in such a way that a current (33) is produced. The kinetic energy increases when the field is switched on. The total kinetic energy  $K$  per unit volume according to Eq. (30) is

$$K = \sum k_i = \frac{1}{2m} \sum p_i^2 + \frac{e^2 n}{2mc^2} A^2 \quad (34)$$

(the  $\vec{p}_i \cdot \vec{A}$  term vanishes because  $P=0$ ). In a non-superconducting metal this increase is compensated within a short relaxation time by changing the momentum  $\vec{p}_i$  of each electron by  $\frac{e\vec{A}}{c}$ , so that the kinetic energy  $K$  reverts to its original value\*\*\*). Then the total current would become zero again. This

\*) Every physicist should know that the lengths  $r_0$ ,  $\lambda_c$ ,  $a_b$ ,  $\lambda_{Ry}$  are all in the ratio 137, where  $\lambda_c$  is the Compton wavelength and  $\lambda_{Ry}$  is the wavelength corresponding to the Rydberg.

\*\*\*) The neon shell does not contain electrons with higher angular momentum than one unit of  $h$ .

\*\*\*\*) Actually not every electron needs to change its momentum in order to obtain that new momentum distribution, but only a few near the top of the Fermi distribution. The ones for which  $\vec{p}_i$  is parallel or nearly parallel to  $\vec{A}$  have increased their energy, and the ones for which  $\vec{p}_i$  is in the opposite direction have decreased it. The new momentum distribution is obtained by reverting the direction of  $\vec{p}_i$  of the former ones.

rearrangement is prevented, however, by the Cooper-pair crust on top of the Fermi distribution. The momentum distribution of the electrons cannot change; it costs a definite finite energy to change it, even by very small amounts<sup>\*)</sup>.

We now sketch the famous conclusion by Fritz London according to which it follows from Eq. (33) that a magnetic field cannot penetrate into a superconductor deeper than the distance  $\lambda$ . This is why  $\lambda$  is called "penetration length". Let us use one of the Maxwell equations, namely  $\text{curl } B = j/c$ , and write Eq. (33) in the following form:

$$\text{curl } B = \frac{-1}{\lambda^2} A .$$

Of course, this equation applies only within a superconductor. We now apply another curl to it ( $\text{curl curl} = -\nabla^2$ ):

$$\nabla^2 B = \frac{1}{\lambda^2} B . \quad (35)$$

This relation shows that, within a superconductor,  $B$  must behave like an exponential  $B \sim e^{\pm x/\lambda}$ , where  $x$ , for example, may be directed into the conductor. Clearly only the minus sign is possible if the thickness of the conductor is much larger than  $\lambda$  (see Fig. 9).

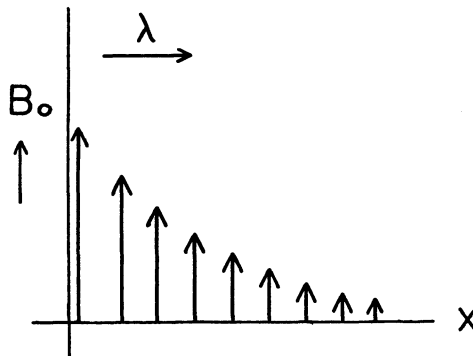


Fig. 9 The penetration of a magnetic field  $B_0$  into a superconductor. The material fills the space  $x > 0$ .

Our final aim is the description of a supercurrent in a straight cylindrical wire of radius  $R$  and practically infinite length. Before doing this, let us consider an infinite superconductor with a plane surface. Imagine that in a rectangular coordinate system the space  $x > 0$  is filled with a superconducting metal; the plane defined by  $x=0$  is its surface. Imagine further

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\*) One may conclude that an insulator should also carry a supercurrent since there is also a gap (even a much larger one) between the filled band of electron states and the next unfilled band. Appendix II tells why an insulator is not a superconductor.



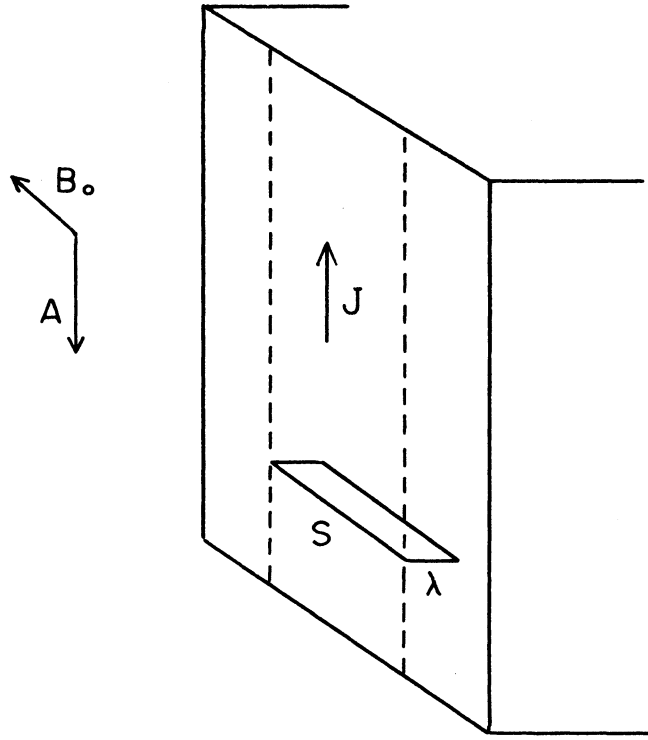


Fig. 10 The surface of a superconductor. A field  $B_0$  in the indicated direction produces a current  $J$  upwards at the surface in a layer of depth  $\lambda$ . The current in a strip of width  $s$  is proportional to  $B_0$ :  $(1/c)J = sB_0$ . The corresponding vector potential  $A$  points downwards.

that outside, in the region  $x < 0$ , there is a magnetic field  $B_0$  in the  $y$  direction (see Fig. 10). We then conclude from Eq. (35) that inside the metal ( $x > 0$ ) the magnetic field must be

$$B_y = B_0 e^{-x/\lambda}, \quad x > 0, \quad (36)$$

and all other components ( $B_z, B_x$ ) vanish. The corresponding vector potential is ( $\text{curl } \vec{A} = \vec{B}$ )

$$A_z = -\lambda B_0 e^{-x/\lambda}, \quad x > 0, \quad (37)$$

with  $A_x = A_y = 0$ . It points downwards in the  $z$ -direction. According to Eq. (33), there must be a current density  $\vec{j}$ :

$$\frac{1}{c}j_z = -\frac{1}{\lambda^2}A_z = \frac{1}{\lambda}B_0 e^{-x/\lambda},$$

with  $j_x = j_y = 0$ . The current density flows upwards in the metal, but only within a depth of  $\lambda$  from the surface. Let us replace the exponential by a step function:

$$e^{-x/\lambda} = \begin{cases} 1 & \text{for } x < \lambda \\ 0 & \text{for } x > \lambda \end{cases}. \quad (38)$$

It is useful to determine the total current  $J_s$  flowing through a rectangle in the (x-y)-plane of length  $s$  in the y-direction and a width  $\lambda$  in the x-direction, lying inside the metal, but adjacent to the surface. Because of our assumption (38), the current density across the rectangle is equal to  $B_0/\lambda$ . Then, the current  $J_s$  is

$$\frac{1}{c}J \cdot s = \frac{B_0}{\lambda} \cdot \lambda \cdot s = s \cdot B_0. \quad (39)$$

This must be interpreted as follows: as long as there is a magnetic field  $B_0$  in the y-direction, outside the surface of a superconductor, there is a current running in the z-direction on the surface of the metal within a depth  $\lambda$ , and this current is  $cB_0$  per cm, measured perpendicular to the current in the y-direction.

Now let us consider the superconducting wire of radius  $R$  with its axis in the z-direction, with a circular magnetic field  $B_0$  around the wire, parallel to its surface and perpendicular to the z-axis. We then conclude that there will be a current  $J$  in the wire flowing in the z-direction of strength

$$J = 2\pi R c B_0. \quad (40)$$

The current is restricted to the surface of the wire within a depth  $\lambda$  (see Fig. 11). We know from the Maxwell equations that the current (40) produces at the radius  $R$  a circular magnetic field identical with  $B_0$ . Thus the situation is self-consistent and stable. The current produces the field  $B_0$ ,

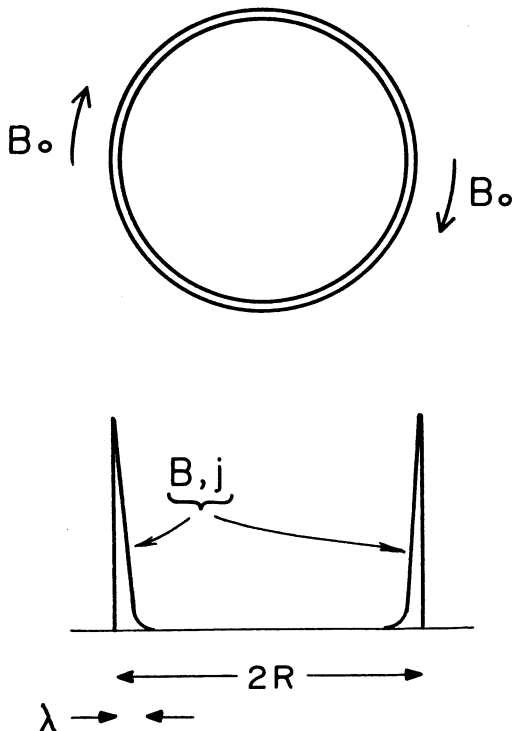


Fig. 11 A cross-section through a superconducting wire of radius  $R$ . The current flows within a thin surface layer  $\lambda$  into which the magnetic field penetrates.

which in turn causes the same current to appear in the "frozen" electron gas, kept in its ground state  $P=0$  by the Cooper-pair crust on the top of Fermi distribution, as long as  $kT < \Delta$ . The electrons that are inside the wire by more than  $\lambda$  do not produce a current; there is no magnetic field there. But at the surface within  $\lambda$ , the electrons move (those in Cooper pairs and those not in pairs) although their total momentum remains zero. This is so because the magnetic field penetrates into the metal to a depth  $\lambda$ . This current is permanent, since no energy is available to break the frozen electron distribution. Dissipation of the kinetic energy of the electrons is excluded.

There is a limit to this situation. As indicated at an earlier stage, the supercurrent Eq. (33) breaks down when the additional kinetic energy which the electrons acquire becomes larger than the energy necessary to break the Cooper pairs. Then it would pay to break the frozen distribution and to produce a new one in which there is no current and consequently no surplus kinetic energy. Let us look at a square centimeter of the surface of a superconductor. The current, and therefore the surplus kinetic energy, is contained within a depth  $\lambda$ , that is in a volume  $V = \lambda \text{ cm}^3$ . The additional kinetic energy  $\Delta K$  in that volume can be determined from the second term in Eq. (34). This term is the energy surplus per unit volume, so that we get

$$\Delta K = V \frac{ne^2}{2mc^2} A^2 = V \frac{1}{2\lambda^2} A^2 . \quad (41)$$

The current will be stable if  $\Delta K$  is less than the total energy  $G$  of the binding of Cooper pairs:

$$G = V \cdot n' \Delta , \quad (42)$$

where  $n' = \frac{\Delta}{\epsilon_F} n$ , the number of Cooper pairs in a  $\text{cm}^3$ . If  $\Delta K$  is larger than  $G$ , it would pay to destroy the Cooper pairs and thus the supercurrent. We get the limiting field  $B_{\text{cr}}$  and a limiting current according to Eq. (33) by equating Eq. (42) with Eq. (41). We will be semiquantitative now, and get

$$n \frac{\Delta}{\epsilon_F} \Delta \sim \frac{1}{\lambda^2} A_{\text{cr}}^2 .$$

From Eq. (36) and Eq. (37) we learn that  $A^2/\lambda^2 = B^2$  and we obtain

$$B_{\text{cr}}^2 \sim \frac{1}{d^3} \left( \frac{d}{\rho} \right)^2 \epsilon_F \sim \left( \frac{e}{d\rho} \right)^2 . \quad (43)$$

Here we used  $n = d^{-3}$ ,  $\rho/d \sim \Delta/\epsilon_F$ ,  $\epsilon_F \sim e^2/d$ . Thus the critical field at which superconductivity breaks down is  $e/(d\rho)$ . It is of the order of 300 G, if  $\Delta \sim 10^{-4} \epsilon_F$ , and  $\epsilon_F \sim 5 \text{ eV}$ .

## APPENDIX I

It would seem at first sight that two electrons on the top of the Fermi distribution that are running almost in the same direction, one behind the other, would also be subject to the full amount of the attractive potential. This is not so. The reasons are as follows. In considering a two-particle problem one first must separate the movement of the center of mass from the relative motion. In this case the momentum  $P$  of the center of mass motion is  $P \sim 2p_F$ . Of course, this motion is not subject to the mutual attraction. The relative motion of the two electrons is subject to the full attraction only if it is parallel to  $\vec{P}$ . If it deviates appreciably from this direction, it will not stay within the potential tube. Hence, the potential of the relative motion is highly asymmetric in the directions; it is a long tube of length  $\ell$  and width  $d$  in the  $\vec{P}$  direction and reaches only to a distance  $d$  in the directions orthogonal to  $\vec{P}$ . Such a potential would lead to a very much smaller binding. A rough estimate, taking the angular average of this potential, would lead to the same formula (23) with factor  $\frac{\ell^2}{d^2} = \beta^2$  in the negative exponent.

APPENDIX II

We may ask the following question: In an insulator the electron distribution fills up a band of electron states from bottom to the top. Thus there is an energy gap between the occupied band and the next unoccupied one. One would expect that this gap has an effect similar to that of the gap above the frozen electron distribution of a superconductor. It also prevents an electron from breaking away from a distribution whose total momentum is zero. Why is not an insulator a perfect superconductor and with a much higher transition temperature since the gap in an insulator is of the order of 1 V?

The answer lies in the fact that the expression (30) for the kinetic energy of an electron is not even approximately correct near the upper end of the band. Let us call  $k(p)$  the kinetic energy of the electron states as function of the momentum in the absence of any fields. Let us for simplicity's sake assume that  $p$  has only one dimension;  $k(p)$  is a quadratic function  $p^2/2m$  only for the lower part of the band. When  $p$  gets near to the upper end,  $p_{\max}$  of the band, the function  $k(p)$  has a point of inflection and has a horizontal tangent at the upper end (see Fig. 12). The velocity is the derivative of  $k$  in respect to  $p$ . Hence the current can be expressed by

$$j = e \sum \frac{dk}{dp} = e \int_{-p_m|a}^{+p_m|a} \frac{dk}{dp} \frac{Ldp}{h} \quad (A1)$$

where, in the second equal sign, the sum over all electrons is replaced by an integral in which  $\frac{Ldp}{h}$  is the number of states in the interval  $dp$ . ( $L$  is the one-dimensional size of the metal.) Clearly (A1) must give zero since  $(dk/dp)$  is opposite and equal for  $p$  and  $-p$ . It shows that the total current is zero when no field is present. If there is a field, the function  $k(p)$  must be replaced by  $k(p - \frac{eA}{c})$ . In general  $\frac{eA}{c} \ll p$ , so that we may express this replacement by a differentiation:

$$k(p - \frac{eA}{c}) = - \frac{eA}{c} \cdot \frac{dk}{dp} . \quad (A2)$$

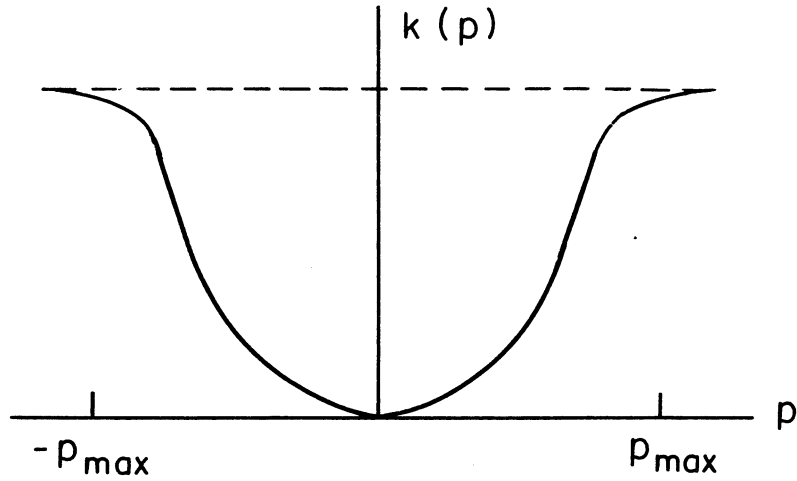


Fig. 12 The kinetic energy  $k(p)$  as function of  $p$  in a conduction band. The derivative is zero at  $\pm p_{\max}$ . In an insulator the band is filled and the top electrons have zero velocity. In a metal it is partially filled, and the top electrons have a velocity  $\sim \pm p/m$ .

Thus we get for the change of current of  $A \neq 0$ :

$$j = -\frac{e^2}{c} A \int_{-p_{\max}}^{+p_{\max}} \frac{d^2 k}{dp^2} \frac{L}{\hbar} dp = \frac{e^2}{c} \frac{L}{\hbar} A \left. \frac{dk}{dp} \right|_{-p_{\max}}^{+p_{\max}} = 0$$

since the derivative of  $k$  vanishes at  $\pm p_{\max}$ . No current is induced by a magnetic field in a completely filled conduction band!

The reason for this remarkable result is this. The presence of a field adds an amount  $-\frac{e}{c}A$  to all momenta  $p$ . Thus the electron distribution is no longer symmetric around  $p=0$ . It amounts to a displacement of all momenta by  $-\frac{e}{c}A$ . Since the distribution was compact (all levels with  $|p| < |p_{\max}|$  occupied), such a displacement is equivalent to transferring a few electrons from the extreme positions at one end (say near  $-p_{\max}$  if  $\frac{e}{c}A$  is negative) to the other end (near  $+p_{\max}$ ). If the band is not filled to the top, such a shift produces a current, since the electrons on one side of  $p=0$  move in opposite directions to those on the other side. If the band is full, however, the velocity of the electrons near  $p_{\max}$  is zero, since  $v = dk/dp = 0$  at the edge of the band. Thus the transfer of electrons from one side to the other does not change the current, which was zero to begin with.