Lecture Notes

These notes are a summary of the main definitions and results of the course. Demonstrations, discussions and methods are found in the lectures and tutorials. Comments and suggestions welcome at laurent.verstraete@u-psud.fr.

I- Relativity postulates and their consequences

In classical physics, objects and processes are described in frames (noted $\mathcal{R}$), the association of a Euclid space with a time measure t. Space is assumed to be isotropic and homogeneous and duration (or time) measures are taken to be the same in all frames. A particular class of frame is defined, the *inertial frames* where the inertia principle (Newton's first law[(a)]) is verified.
First stated by Galileo in 1632, the *relativity principle* is now one of the corner stones of physics. Its modern form is: the laws of physics take the same form in all inertial frames. By the end of the 19th century, it was soon realized that electromagnetism did not fulfill the relativity principle. At the same time, the Michelson and Morley experiment suggested that the speed of light in free space was the same in all frames This lead Einstein to the *postulates of Special relativity* namely, (1) the principle of relativity is verified and (2) the speed of light $c \simeq 3\,10^8$ m/s is the same in all inertial frames.
It was then shown that the application of these postulates to the transformation of events between frames $\mathcal{R}$ and $\mathcal{R}'$, in rectilinear uniform motion with respect to each other, lead to the Lorentz-Poincaré transformation, already known in the context of electromagnetism (1899)

$$\begin{cases} ct' = \gamma(ct - \beta x) \\ x' = \gamma(x - \beta ct) \\ y' = y \\ z' = z \end{cases}$$

with $\beta = u/c$, $\gamma = 1/\sqrt{1-\beta^2}$ and $\overrightarrow{\mathrm{u}}$ the velocity of $\mathcal{R}'$ with respect to $\mathcal{R}$. The Lorentz transformation has several important consequences:

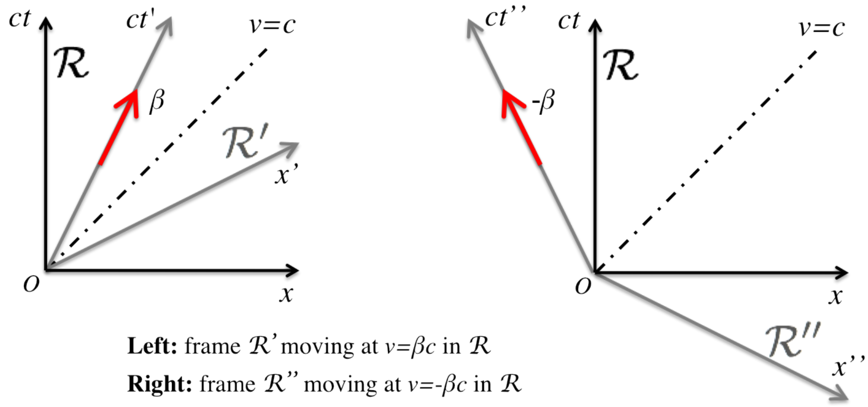
- events simultaneous in $\mathcal{R}$ are not simultaneous anymore in $\mathcal{R}'$. In relativity, time and space are linked and physics must be described in a space with 4 dimensions called *spacetime*. Points in this 4D space are called *events*. The spacetime event $(t, \overrightarrow{\mathrm{r}})$ in $\mathcal{R}$ thus takes place at $(t', \overrightarrow{\mathrm{r}}')$ in $\mathcal{R}'$.

- The quantity $s^2 = (ct)^2 - \overrightarrow{\mathrm{r}}^2 = (ct')^2 - \overrightarrow{\mathrm{r}}'^2$ is conserved between frames[(b)]: it is a spacetime squared distance or interval. The squared interval between events A and B is $s_{AB}^2 = c^2(t_A - t_B)^2 - (\overrightarrow{\mathrm{r_B}} - \overrightarrow{\mathrm{r_A}})^2$ or, in infinitesimal form, $\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \overrightarrow{\mathrm{d}r}^2$. By definition a length L is the spacetime interval between two simultaneous events here $L = |\overrightarrow{\mathrm{r_B}} - \overrightarrow{\mathrm{r_A}}|$.

- *Proper frame and related quantities*: by definition the proper frame is the frame where the particle is at rest. A duration (or time) measured in this frame is called *proper time*. Similarly a length measured simultaneously between 2 points in the proper frame is called a *proper length*. If the particle is at rest in $\mathcal{R}'$, t' is the proper time, $\mathrm{d}x' = 0$ and the conservation of ds^2 leads to $\mathrm{d}t = \gamma \mathrm{d}t'$ illustrating the *time dilation* between $\mathcal{R}$ and $\mathcal{R}'$. Similarly a proper length L' measured in $\mathcal{R}'$ will contract when measured from the moving frame $\mathcal{R}$ such that, $L = L'/\gamma$.

Particle or system trajectories in spacetime are represented as world lines in spacetime diagrams. Reference axis of several frames can be represented and moving frames $\mathcal{R}'$, $\mathcal{R}''$ have oblique axis in the lab frame $\mathcal{R}$.

[(a)] In an inertial frame, an object, upon which no force is exerted, moves at constant velocity.
[(b)] Assuming a common origin event $(0, 0)$ in $\mathcal{R}$ and $\mathcal{R}'$



Left: frame $\mathcal{R}$'moving at v=βc in $\mathcal{R}$
Right: frame $\mathcal{R}$'' moving at v=-βc in $\mathcal{R}$
Thick arrow shows the motion of a particle in its proper frame

II- Spacetime physics

In this 4-dimensional space called Minkowski space, vectors are defined as quadruplets $\underline{\mathrm{X}} = (X^\mu)_{\mu=0 \text{ to } 3}$ also called *4-vectors* . In addition, a dot (or scalar) product between 4-vectors $\underline{\mathrm{X}}$ and $\underline{\mathrm{Y}}$ is defined as:

$$\underline{\mathrm{X}}.\underline{\mathrm{Y}} = X^0Y^0 - X^1Y^1 - X^2Y^2 - X^3Y^3 = g_{\mu\nu}\, X^\mu Y^\nu = {}^tX\, G\, Y$$

where $G = (g_{\mu\nu})_{\mu\nu}$ is a 4×4 matrix equal to diag $(1, -1, -1, -1)^{(c)}$. Since the dot product is used to define distances and angles, $g_{\mu\nu}$ will be called the *space metrics.* We note that G is unitary $(G^{-1} = G)$. The Lorentz transformation is represented by a Λ matrix such that $X' = \Lambda\, X$ or $X'^\mu = \Lambda^\mu{}_\nu\, X^\nu$ and conversely $X^\mu = \Lambda_\mu{}^\nu\, X'^\nu$ with $\Lambda_\mu{}^\nu = \Lambda^{-1} = \Lambda(-\beta)$. The dot product provides *4-scalars* which are conserved through Lorentz transformations and called *Lorentz invariants*

A physical 4-vector $\underline{\mathrm{A}}$ is defined as a quadruplet whose norm is a Lorentz invariant and whose coordinates transform according to the Lorentz transformation, $A' = \Lambda A$ from $\mathcal{R}$ to $\mathcal{R}'$. Important 4-vectors are

- 4-position: $\underline{\mathrm{X}} = (ct, \overrightarrow{\mathrm{r}})$ of norm $c^2t^2 - \overrightarrow{\mathrm{r}}^2$
- 4-velocity: $\underline{\mathrm{U}} = \dfrac{\mathrm{d}\underline{\mathrm{X}}}{\mathrm{d}\tau} = \gamma(c, \overrightarrow{\mathrm{v}})$ with norm c^2,
- 4-momentum: $\underline{\mathrm{P}} = m\underline{\mathrm{U}}$ of norm $m^2\, c^2$. It is also written as $\underline{\mathrm{P}} = (E/c, \overrightarrow{\mathrm{p}})$ with $E = \gamma mc^2$ and $\overrightarrow{\mathrm{p}} = \gamma m \overrightarrow{\mathrm{v}}$ the relativistic total energy and momentum. Defining the *kinetic energy* as $K = (\gamma - 1)mc^2$ we have $E = K + mc^2$ with mc^2 the *mass energy.* From the norm of $\underline{\mathrm{P}}$ we also obtain the relationship $E^2 = p^2c^2 + m^2c^4$.
- 4-force and acceleration: defined as $\underline{\mathrm{F}} = \dfrac{\mathrm{d}\underline{\mathrm{P}}}{\mathrm{d}\tau}$ or $\underline{\mathrm{F}} = \gamma\left(\dfrac{1}{c}\dfrac{\mathrm{d}E}{\mathrm{d}t}, \dfrac{\mathrm{d}\overrightarrow{\mathrm{p}}}{\mathrm{d}t}\right)$ with $\overrightarrow{\mathrm{p}} = \gamma m \overrightarrow{\mathrm{v}}$.

  We have the property $\underline{\mathrm{F}}.\underline{\mathrm{U}} = 0$ hence $\dfrac{\mathrm{d}K}{\mathrm{d}t} = \overrightarrow{\mathrm{f}}.\overrightarrow{\mathrm{v}}$ with $\overrightarrow{\mathrm{f}} = \dfrac{\mathrm{d}\overrightarrow{\mathrm{p}}}{\mathrm{d}t}$, the relativistic dynamical principle. The 4-acceleration is defined as $\underline{\Gamma} = \dfrac{\underline{\mathrm{F}}}{m} = \dfrac{\mathrm{d}\underline{\mathrm{U}}}{\mathrm{d}\tau}$.
- 4-wave vector: $\underline{\mathrm{K}} = \left(\dfrac{\omega}{c}, \overrightarrow{\mathrm{k}}\right)$ for a wave of angular velocity ω and wave vector $\overrightarrow{\mathrm{k}}$. In free space $\underline{\mathrm{K}}^2 = 0$. The wave phase $\varphi = \omega t - \overrightarrow{\mathrm{k}}.\overrightarrow{\mathrm{r}} = \underline{\mathrm{K}}.\underline{\mathrm{X}}$ is thus invariant (4-scalar).

- *Doppler effect:* the formula is easily established from the transformation of the time part of $\underline{\mathrm{K}}$, namely $\omega' = \gamma\left(\omega - \overrightarrow{\mathrm{u}}.\overrightarrow{\mathrm{k}}\right)$ where the emitted frequency is ω in $\mathcal{R}$ and detected as ω' in $\mathcal{R}'$.

When *physical laws* are written in the Minkowski space using spacetime coordinates and 4-vectors they can be transformed from one frame to the other or to find Lorentz invariants.

$^{(c)}$The notation $g_{\mu\nu}\, A^\mu B^\nu$ uses the Einstein summation rule that applies on indices placed up and down.

III- Relativistic Dynamics

Important applications of special relativity are found in collisions of free particles such as: $A + B \rightarrow C + D$ where the 4-momentum is conserved, $\underline{\mathrm{P}_A} + \underline{\mathrm{P}_B} = \underline{\mathrm{P}_C} + \underline{\mathrm{P}_D}$ ensuring conservation of total energy and momentum.

- *Center of mass frame:* we note $M = \sum_i m_i$ with $i = A, B$ or $i = C, D$ from our example of collision above. By definition, the center of mass frame or center of momentum frame (CMF) $\mathcal{R}^*$ is the frame in which the total momentum of particles is zero: $\overrightarrow{\mathrm{p}}^* = \sum_i \overrightarrow{\mathrm{p}_i}^* = \overrightarrow{0}$. Since momentum is conserved in the collision the CMF is the same for incoming (A, B) and outcoming (C, D) particles. The total momentum of the particles in $\mathcal{R}$ is $\overrightarrow{\mathrm{p}} = \sum_i \overrightarrow{\mathrm{p}_i}$ and can be expressed with a Lorentz transformation of $\overrightarrow{\mathrm{p}}^*$. The velocity $\overrightarrow{\mathrm{v}_G}$[(d)] of $\mathcal{R}^*$ with respect to $\mathcal{R}$ is found from the definition $\overrightarrow{\mathrm{p}} = \gamma_G M \overrightarrow{\mathrm{v}_G}$, i.e., $\overrightarrow{\mathrm{v}_G} = \frac{c^2}{E} \overrightarrow{\mathrm{p}_G}$ with $\gamma_G = E/E^*$ and $E^* = Mc^2$.

- *Elastic collision:* in this case the kinetic energy K is also conserved. Since the total energy is $E = K + mc^2$ this implies that the mass energy is conserved or $m_A + m_B = m_C + m_D$ in the above case.

- *The relativistic dynamical principle* writes in $\mathcal{R}$ as $\dfrac{\mathrm{d}\overrightarrow{\mathrm{p}}}{\mathrm{d}t} = \overrightarrow{\mathrm{f}}$ where $\overrightarrow{\mathrm{f}}$ is the resulting force on the particle and $\overrightarrow{\mathrm{p}} = \gamma m \overrightarrow{\mathrm{v}}$. In the case of the Lorentz force on a particle of charge q we have $\overrightarrow{\mathrm{f}} = q(\overrightarrow{\mathrm{E}} + \overrightarrow{\mathrm{v}} \times \overrightarrow{\mathrm{B}})$ and the dynamical principal can be written in terms of 4-vectors: $\dfrac{\mathrm{d}\underline{\mathrm{P}}}{\mathrm{d}\tau} = q\, F\, \underline{\mathrm{U}}$ where F is the electromagnetic tensor (see below).

IV- Relativistic Electrodynamics

From charge conservation and the fact that the spacetime element $\mathrm{d}\mathbb{V} = c\mathrm{d}t\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$ is a Lorentz invariant, other 4-vectors can be defined

- 4-current density: $\underline{\mathrm{J}} = (\rho c, \overrightarrow{\mathrm{j}} = \rho \overrightarrow{\mathrm{v}}) = \dfrac{\rho}{\gamma}\, \underline{\mathrm{U}}$
- 4-potentiel: $\underline{\mathrm{A}} = \left(\dfrac{\Phi}{c}, \overrightarrow{\mathrm{A}}\right)$.
- 4-derivation: $\underline{\nabla} = \left(\dfrac{1}{c}\dfrac{\partial}{\partial t}, -\overrightarrow{\nabla}\right)$ of norm $\Box = \dfrac{1}{c^2}\partial t^2 - \overrightarrow{\nabla}^2$ (d'Alembertian operator).

A Lorentz invariant formulation of electromagnetism is obtained if the $\overrightarrow{\mathrm{E}}$ and $\overrightarrow{\mathrm{B}}$ fields are gathered in a matrix F called the *electromagnetic tensor*

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

which provides two Lorentz invariants $F^{\mu\nu}F_{\mu\nu} = 2(B^2 - E^2/c^2)$ and $\det(F) = (\overrightarrow{\mathrm{E}}.\overrightarrow{\mathrm{B}})^2/c^2$.

From frame $\mathcal{R}$ to $\mathcal{R}'$, the electromagnetic tensor transforms as $F' = \Lambda\, F\, \Lambda$ if the fields transform as

$$\begin{aligned} \overrightarrow{\mathrm{E}'} &= \overrightarrow{\mathrm{E}_\parallel} + \gamma\overrightarrow{\mathrm{E}_\perp} + \overrightarrow{\mathrm{U}} \times \overrightarrow{\mathrm{B}} \\ \overrightarrow{\mathrm{B}'} &= \overrightarrow{\mathrm{B}_\parallel} + \gamma\overrightarrow{\mathrm{B}_\perp} - \overrightarrow{\mathrm{U}} \times \overrightarrow{\mathrm{E}}/c^2 \end{aligned}$$

with $\overrightarrow{\mathrm{U}} = \gamma\overrightarrow{\mathrm{v}}$ and where the $\parallel$ field components are colinear to the direction of motion of $\mathcal{R}'$ with respect to $\mathcal{R}$, the x-axis here (and the $\perp$ components are perpendicular to the x-axis).

Maxwell source equations are then obtained from $\underline{\nabla}^2\underline{\mathrm{A}} = \underline{\nabla}F = \mu_0\, \underline{\mathrm{J}}$ if a Lorenz gauge is adopted, $\underline{\nabla}.\underline{\mathrm{A}}$=0.

[(d)] This is the velocity of the center of mass G, the origin of $\mathcal{R}^*$. In the case of free particles, G is found from $\gamma_G M\, \overrightarrow{OG} = \sum_i \gamma_i m_i\, \overrightarrow{OM_i}$.