Rayon lumineux dans un milieu d'indice variable

On calcule le rayon lumineux en résolvant numériquement, par la méthode de Runge-Kutta (voir formules page suivante), l'équation:

$$\frac{d}{ds}(n\vec{t}) = \vec{\nabla} n$$

$\vec{t}$ étant le vecteur unitaire tangent au rayon lumineux au point $M(x, y, z)$ d'abscisse curviligne s, défini par:

$$\vec{t} = \frac{d\vec{OM}}{ds}$$

On applique ensuite à un ou plusieurs des exemples suivants:

1) Fibre optique

Cylindre de révolution dont l'indice varie suivant la loi:

$$n(r) = n_0(1 - \frac{r^2}{2\rho^2})$$

n_0 valeur de l'indice sur l'axe, ρ est un paramètre.

2) Milieu soumis à une onde acoustique

$$n(x) = n_0 + n_1 \cos \frac{2\pi}{\Lambda} x$$

Λ longueur d'onde de l'onde acoustique, $n_1 < n_0$, les plans x =constante ont même indice.

3) Mirages

$$n(x) = n_0 + \alpha x$$

ou $$n(x) = n_0(1 + \tfrac{2x}{a})^{\frac{1}{2}}$$

4) Œil de poisson de Maxwell

Voir pages suivantes.

Références:

J. Ph. Pérez Optique p. 157
Born Wolf Principles of Optics p. 147

$$\frac{d}{ds}\left(n\vec{t}\right) = \vec{\nabla} n \qquad \vec{t} = \frac{d\vec{OM}}{ds} \qquad \vec{OM}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\frac{dn}{ds} = \frac{\partial n}{\partial x}\frac{dx}{ds} + \frac{\partial n}{\partial y}\frac{dy}{ds} + \frac{\partial n}{\partial z}\frac{dz}{ds}$$

$$\frac{dn}{ds}\frac{d\vec{OM}}{ds} + n\frac{d^2\vec{OM}}{ds^2} = \vec{\nabla} n$$

$$\frac{d^2\vec{OM}}{ds^2} = \frac{1}{n}\left[\vec{\nabla} n - \frac{dn}{ds}\frac{d\vec{OM}}{ds}\right]$$

On pose $\frac{d\vec{OM}}{ds} = \vec{V}$ $\qquad \vec{V}\begin{pmatrix} \frac{dx}{ds} \\ \frac{dy}{ds} \\ \frac{dz}{ds} \end{pmatrix}$ ← ça sert à quoi ?

$$\frac{d\vec{V}}{ds} = \frac{1}{n}\left[\vec{\nabla} n - \frac{dn}{ds}\vec{V}\right]$$

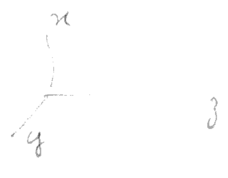
On pose $q[0] = x \quad q[1] = y \quad q[2] = z \quad q[3] = \frac{dx}{ds} \quad q[4] = \frac{dy}{ds} \quad q[5] = \frac{dz}{ds}$

Donc :

$$\dot{q}[0] = q[3]$$
$$\dot{q}[1] = q[4]$$
$$\dot{q}[2] = q[5]$$
$$\dot{q}[3] = \frac{1}{n}\left(\frac{\partial n}{\partial x} - \frac{dn}{ds}q[3]\right)$$
$$\dot{q}[4] = \frac{1}{n}\left(\frac{\partial n}{\partial y} - \frac{dn}{ds}q[4]\right)$$
$$\dot{q}(5) = \frac{1}{n}\left(\frac{\partial n}{\partial z} - \frac{dn}{ds}q[5]\right)$$

$$\frac{dn}{ds} = \frac{\partial n}{\partial x}q[3] + \frac{\partial n}{\partial y}q[4] + \frac{\partial n}{\partial z}q[5]$$

1) Fibre optique $\qquad n = n_0\left(1 - \frac{r^2}{2\rho^2}\right)$



$r^2 = x^2 + y^2$

$$\frac{\partial r^2}{\partial x} = 2x \qquad \frac{\partial (r^2)}{\partial y} = 2y \qquad \frac{\partial (r^2)}{\partial z} = 0$$

$$\frac{\partial n}{\partial x} = -n_0\frac{x}{\rho^2} \qquad \frac{\partial n}{\partial y} = -n_0\frac{y}{\rho^2} \qquad \frac{\partial n}{\partial z} = 0$$

2) Onde acoustique : $n = n_0 + \Delta n_0 \sin 2\pi \frac{x}{\Lambda}$

$$\frac{\partial n}{\partial x} = \Delta n_0 \frac{2\pi}{\Lambda} \cos 2\pi \frac{x}{\Lambda} \qquad \frac{\partial n}{\partial y} = 0 \qquad \frac{\partial n}{\partial z} = 0$$

3) Atmosphère terrestre : $h_0 = 8\,10^3\ m$ $\Delta n = 2.77\,10^{-4}$

$$n = 1 + \Delta n\, e^{-\frac{r - r_0}{h_0}}$$

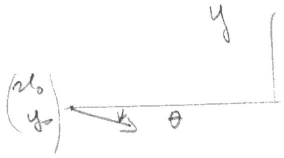
→ indice de l'air à la surface de la Terre

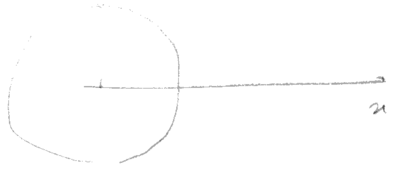
$n_0 = 1.000277$

$h = r - r_0$

r_0 rayon de la terre

On se place dans un plan





$$\frac{\partial n}{\partial x} = \frac{dn}{dr}\frac{\partial r}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{\partial \sqrt{x^2 + y^2 + z^2}}{\partial x} = \frac{2x}{2r} = \frac{x}{r}$$

$$\left| \begin{array}{l} \ddot{x} = \frac{1}{n}\left(\frac{\partial n}{\partial x} - \frac{dn}{ds}\dot{x}\right) \\ \ddot{y} = \frac{1}{n}\left(\frac{\partial n}{\partial y} - \frac{dn}{ds}\dot{y}\right) \end{array} \right.$$

$$\frac{dn}{ds} = \frac{\partial n}{\partial x}\frac{dx}{ds} + \frac{\partial n}{\partial y}\frac{dy}{ds} = \frac{dn}{dr}\left(\frac{\partial r}{\partial x}\dot{x} + \frac{\partial r}{\partial y}\dot{y}\right) = \frac{dn}{dr}\frac{1}{r}(x\dot{x} + y\dot{y})$$

$$\left| \begin{array}{l} \ddot{x} = \frac{1}{n}\frac{dn}{dr}\frac{1}{r}\left[x - \dot{x}(x\dot{x} + y\dot{y})\right] \\ \ddot{y} = \frac{1}{n}\frac{dn}{dr}\frac{1}{r}\left[y - \dot{y}(x\dot{x} + y\dot{y})\right] \end{array} \right. \qquad \left| \begin{array}{l} \ddot{x} = f(r)\left[x - \dot{x}(x\dot{x} + y\dot{y})\right] \\ \ddot{y} = f(r)\left[y - \dot{y}(x\dot{x} + y\dot{y})\right] \end{array} \right.$$

$$\frac{dn}{dr} = -\frac{\Delta n}{h_0} e^{-\frac{(r - r_0)}{h_0}} \qquad \frac{1}{n}\frac{dn}{dr} = \frac{-\Delta n / h_0}{e^{\frac{r - r_0}{h_0}} + \Delta n}$$

$$\Rightarrow \boxed{\frac{1}{nr}\frac{dn}{dr} = \frac{-1}{r h_0 \left(\frac{e^{\frac{r - r_0}{h_0}}}{\Delta n} + 1\right)} = f(r)}$$

Conditions initiales :

$x(0) = x_0$ $y(0) = y_0$ $\frac{dx}{ds}(0) = \cos\theta$ $\frac{dy}{ds}(0) = \sin\theta$

4.2.2 Maxwell's "fish-eye"

A simple and interesting example of an absolute instrument is presented by the medium which is characterized by the refractive index function

$$n(r) = \frac{1}{1 + (r/a)^2} n_0, \tag{14}$$

where r denotes the distance from a fixed point O, and n_0 and a are constants. It is known as the "fish-eye" and was first investigated by MAXWELL.†

It was shown in § 3.2 that in a medium with spherical symmetry the rays are plane curves which lie in planes through the origin, and that the equation of the rays may be written in the form (cf. § 3.2 (11))

$$\theta = c \int^r \frac{dr}{r\sqrt{n^2(r)r^2 - c^2}},$$

c being a constant. On substituting from (14) and setting

$$\rho = \frac{r}{a}, \qquad K = \frac{c}{an_0}, \tag{15}$$

† J. C. MAXWELL, *Cambridge and Dublin Math. J.*, **8** (1854), 188; also *Scientific Papers*, I (Cambridge University Press), p. 76.

Interesting generalizations of MAXWELL's fish-eye were found by W. LENZ, contribution in *Probleme der Modernen Physik*, edited by P. DEBYE (Leipzig, Hirzel, 1928), 198 and R. STETTLER, *Optik*, **12** (1955), 529. The latter paper also includes a generalization of the so-called *Luneburg lens* which, because of its wide angle scanning capabilities, has useful applications in microwave antenna design. This lens, first considered by R. K. LUNEBURG in his *Mathematical Theory of Optics* (mimeographed lecture notes, Brown University, Providence, R.I., 1944; printed version published by University of California Press, Berkeley and Los Angeles, 1964, § 29), is an inhomogeneous sphere with the refractive index function $n(r) = \sqrt{2 - r^2}$ $(0 \leqslant r \leqslant 1)$. When placed in a homogeneous medium of unit refractive index, it brings to a sharp focus every incident pencil of parallel rays. See also R. F. RINEHART, *J. Appl. Phys.*, **19** (1948), 860; A. FLETCHER, T. MURPHY and A. YOUNG, *Proc. Roy. Soc.*, A **223** (1954), 216; and G. TORALDO DI FRANCIA, *Optica Acta*, **1** (1954–1955), 157.

we obtain

$$\theta = \int^{\rho} \frac{K(1+\rho^2)d\rho}{\rho\sqrt{\rho^2 - K^2(1+\rho^2)^2}}. \tag{16}$$

It may be verified that

$$\frac{K(1+\rho^2)}{\rho\sqrt{\rho^2 - K^2(1+\rho^2)^2}} = \frac{d}{d\rho}\left[\text{arc sin}\left(\frac{K}{\sqrt{1-4K^2}}\frac{\rho^2-1}{\rho}\right)\right],$$

so that (16) becomes

$$\sin(\theta - \alpha) = \frac{c}{\sqrt{a^2 n_0^2 - 4c^2}}\frac{r^2 - a^2}{ar}, \tag{17}$$

where α is a constant of integration.

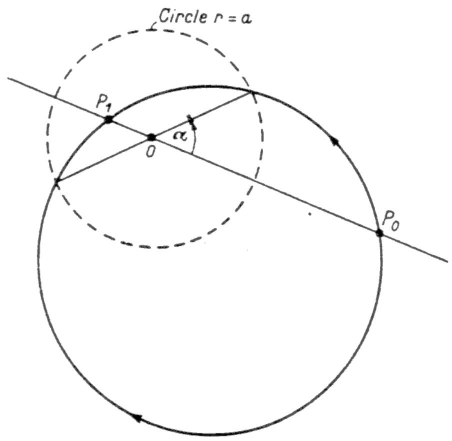


Fig. 4.8. Rays in MAXWELL's "fish-eye".

(17) is the polar equation of the rays. The one-parameter family of rays through a fixed point $P_0(r_0, \theta_0)$ is, therefore, given by

$$\frac{r^2 - a^2}{r\sin(\theta - \alpha)} = \frac{r_0^2 - a^2}{r_0 \sin(\theta_0 - \alpha)}. \tag{18}$$

It is seen that whatever the value of α, this equation is satisfied by $r = r_1$, $\theta = \theta_1$, where

$$r_1 = \frac{a^2}{r_0}, \qquad \theta_1 = \pi + \theta_0, \tag{19}$$

showing that *all the rays from an arbitrary point P_0 meet in a point P_1 on the line joining P_0 to O; P_0 and P_1 are on opposite sides of O and $OP_0 \,.\, OP_1 = a^2$.* Hence the fish-eye is an absolute instrument in which the imaging is an *inversion*.

We note that (17) is satisfied by $r = a$, $\theta = \alpha$ and $r = a$, $\theta = \pi + \alpha$; each ray therefore intersects the fixed circle $r = a$ in diametrically opposite points (see Fig. 4.8).

To obtain the equation of the rays in Cartesian coordinates, we put $x = r\cos\theta$, $y = r\sin\theta$ in (17), and find

$$y\cos\alpha - x\sin\alpha = \frac{c}{a\sqrt{a^2 n_0^2 - 4c^2}}(x^2 + y^2 - a^2),$$

or

$$(x + b\sin\alpha)^2 + (y - b\cos\alpha)^2 = a^2 + b^2 \tag{20}$$

where

$$b = \frac{a}{2c}\sqrt{a^2 n_0^2 - 4c^2}.$$

(20) shows that each ray is a circle.