Follow the Sturmian rabbit

Decidability in the substitutive model

March 8th, 2023

Pierre BÉAUR (supervision: Benjamin HELLOUIN and Nathalie AUBRUN)

Welcome to the Real Word

- in *The Matrix*, the whole universe is encoded with words

⇒ how? and how complex?

- answer: study symbolic dynamics!

What is symbolic dynamics?

- **symbolic**: with symbols (=letters in alphabet $\mathcal{A}$)
- **dynamics**: systems evolving step by step in space

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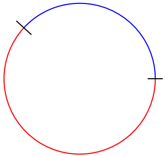

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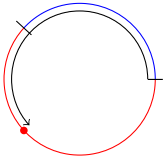

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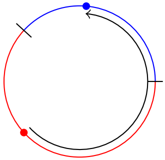

$a\ b$

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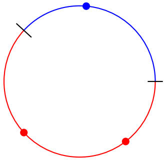

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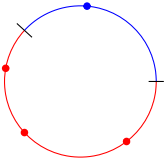

$\color{red}{a}\ \color{blue}{b}\ \color{red}{a}\ \color{red}{a}$

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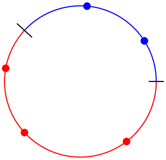

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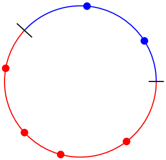

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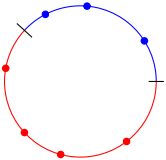

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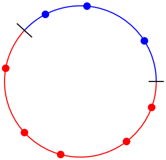

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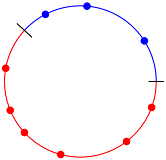

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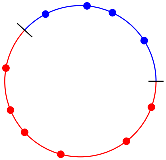

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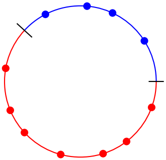

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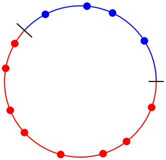

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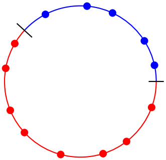

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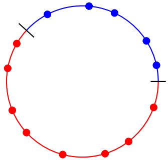
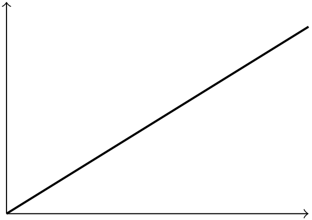


crossings of a line



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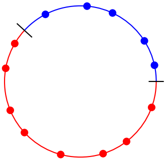

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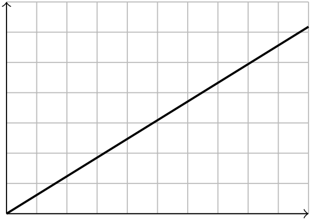

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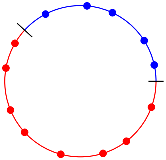

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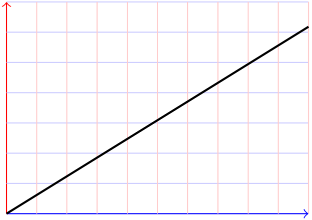

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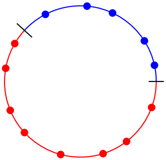

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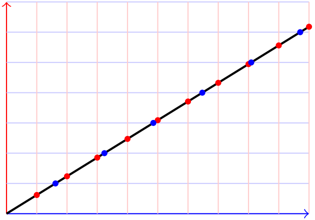

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Now: complexity of an infinite word

- some examples:
  - $aaaaaaaaaaaaaaaaaaaaaaaaa\ldots$ (constant)
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$p_w(n)$ = # of **different factors** of length n in w (always $\leq |\mathcal{A}|^n$)

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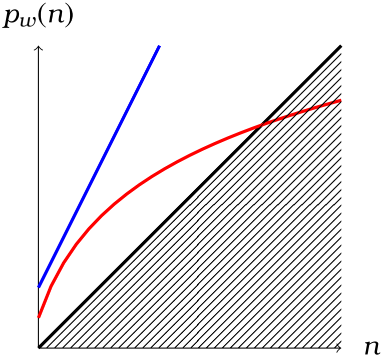

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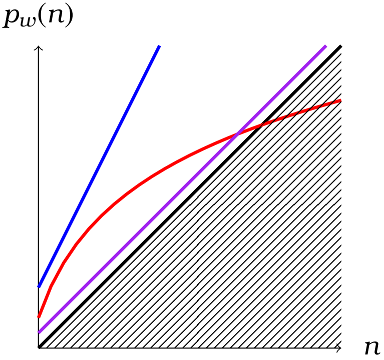

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Combinatorics and dynamics agree, for once

- what are the "simplest" aperiodic words?

Sturmian words ♡

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- ok, but: construction?

A modern method to build words (Axel Thue, 1912)

- new way to build words: substitutions

Substitution

A morphism from letters to words.

Example: $\phi(a) = ab, \phi(b) = a$.

$\Rightarrow \ \phi(abaab) = \phi(a)\phi(b)\phi(a)\phi(a)\phi(b) = ab\ a\ ab\ ab\ a = abaababa$

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Substitutive word

$$\sigma^\infty(a) = \lim_{n\to\infty} \sigma^n(a).$$

Fibo is substitutive!

State-of-the-art on decidability

- a substitutive word: describable to a machine

$\Rightarrow$ can make algorithms to answer questions on them

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Answered questions (Durand 1998-2003)

There are algorithms to decide whether a substitutive word:

- is periodic, ult. periodic, recurrent, unif. recurrent
- is equal to another substitutive word
- has a linear, loglinear, quadratic complexity

- great!

State-of-the-art on decidability

- a substitutive word: describable to a machine

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Answered questions (Durand 1998-2003)

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- has a linear, loglinear, quadratic complexity

- great!
- . . . it works for Sturmian words, right?

Nope.

Sturmian words are (most often) not substitutive.

Hi Mom, I'm on TV!

- what to do for Sturmian words?
- idea: using *multiple* substitutions

$\Rightarrow$ S-adic representations

S-adic representation of Sturmian words

w is Sturmian iff $w = \lim_{n\to\infty} \sigma_0 \circ \sigma_1 \circ \cdots \circ \sigma_n(0)$ (with $\sigma_i \in \{L_0, L_1, R_0, R_1\}$) (*).

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$\Rightarrow$ S-adic representations

S-adic representation of Sturmian words

w is Sturmian iff $w = \lim_{n\to\infty} \sigma_0 \circ \sigma_1 \circ \cdots \circ \sigma_n(0)$ (with $\sigma_i \in \{L_0, L_1, R_0, R_1\}$) (*).

- in literature: mostly seen for math
- no combinatorial exploitation. . . until I come!

My thesis subject

Finding algorithms to decide properties of Sturmian words, and more generally of families of words described by their S-adic representation.

Thank you for your attention!

(There are chocolate thingies in the cafeteria.)